## 4.6 NEWTON'S METHOD

A Click here for answers.

S Click here for solutions.

**1–4** ▪ Use Newton's method with the specified initial approximation $x_1$ to find $x_3$, the third approximation to the root of the given equation. (Give your answer to four decimal places.)

**1.** $x^3 + x + 1 = 0, \quad x_1 = -1$

**2.** $x^7 - 100 = 0, \quad x_1 = 2$

**3.** $x^3 + x^2 + 2 = 0, \quad x_1 = -2$

**4.** $x^5 - 10 = 0, \quad x_1 = 1.5$

**5–6** ▪ Use Newton's method to approximate the given number correct to eight decimal places.

**5.** $\sqrt[4]{22}$

**6.** $\sqrt[10]{100}$

**7–11** ▪ Use Newton's method to approximate the indicated root of the equation correct to six decimal places.

**7.** The root of $x^3 - 2x - 1 = 0$ in the interval $[1, 2]$

**8.** The root of $x^3 + x^2 + x - 2 = 0$ in the interval $[0, 1]$

**9.** The root of $x^4 + x^3 - 22x^2 - 2x + 41 = 0$ in the interval $[3, 4]$

**10.** The positive root of $2 \sin x = x$

**11.** The root of $\tan x = x$ in the interval $(\pi/2, 3\pi/2)$

**12–17** ▪ Use Newton's method to find all roots of the equation correct to six decimal places.

**12.** $x^5 = 5x - 2$

**13.** $x^4 = 1 + x - x^2$

**14.** $(x - 2)^4 = x/2$

**15.** $x^3 = 4x - 1$

**16.** $2 \cos x = 2 - x$

**17.** $\sin \pi x = x$

**18–21** ▪ Use Newton's method to find all the roots of the equation correct to eight decimal places. Start by drawing a graph to find initial approximations.

**18.** $x^4 + 3x^3 - x - 10 = 0$

**19.** $x^9 - x^6 + 2x^4 + 5x - 14 = 0$

**20.** $\sqrt{x^2 - x + 1} = 2 \sin \pi x$

**21.** $\cos(x^2 + 1) = x^3$

## 4.6 ANSWERS

E Click here for exercises. S Click here for solutions.

1. $-0.6860$
2. $1.9308$
3. $-1.6978$
4. $1.5850$
5. $2.16573677$
6. $1.58489319$
7. $1.618034$
8. $0.810536$
9. $3.992020$
10. $1.895494$
11. $4.493409$
12. $-1.582036$, $0.402102$, $1.371882$.
13. $1$, $-0.569840$
14. $1.132529$, $3.117349$
15. $-2.114908$, $0.254102$, and $1.860806$
16. $0$, $1.109144$, $3.698154$
17. $0$, $0.736484$, $-0.736484$
18. $-3.20614267$, $1.37506470$
19. $1.23571742$
20. $0.15438500$, $0.84561500$
21. $0.59698777$

## 4.6 SOLUTIONS

E Click here for exercises.

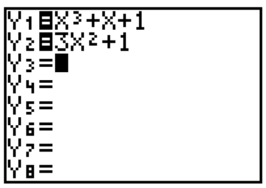
**1.** $f(x) = x^3 + x + 1 \quad \Rightarrow \quad f'(x) = 3x^2 + 1$,

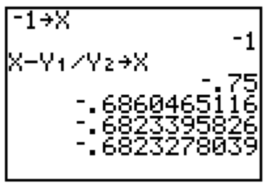
so $x_{n+1} = x_n - \dfrac{x_n^3 + x_n + 1}{3x_n^2 + 1}$. $x_1 = -1$

$\Rightarrow \quad x_2 = -1 - \dfrac{-1 - 1 + 1}{3 \cdot 1 + 1} = -0.75 \quad \Rightarrow$

$x_3 = -0.75 - \dfrac{(-0.75)^3 - 0.75 + 1}{3(-0.75)^2 + 1} \approx -0.6860$.

Here is a quick and easy method for finding the iterations on a programmable calculator. (The screens shown are from the TI-82, but the method is similar on other calculators.) Assign $x^3 + x + 1$ to $\mathrm{Y}_1$ and $3x^2 + 1$ to $\mathrm{Y}_2$. Now store $-1$ in X and then enter $\mathrm{X} - \mathrm{Y}_1/\mathrm{Y}_2 \to \mathrm{X}$ to get $-0.75$. By successively pressing the ENTER key, you get the approximations $x_1$, $x_2$, $x_3$, ....




**2.** $f(x) = x^7 - 100 \quad \Rightarrow \quad f'(x) = 7x^6$, so

$x_{n+1} = x_n - \dfrac{x_n^7 - 100}{7x_n^6}$. $x_1 = 2$

$\Rightarrow \quad x_2 = 2 - \dfrac{128 - 100}{7 \cdot 64} = 1.9375 \quad \Rightarrow$

$x_3 = 1.9375 - \dfrac{(1.9375)^7 - 100}{7(1.9375)^6} \approx 1.9308$.

**3.** $f(x) = x^3 + x^2 + 2 = 0 \quad \Rightarrow \quad f'(x) = 3x^2 + 2x$,

so $x_{n+1} = x_n - \dfrac{x_n^3 + x_n^2 + 2}{3x_n^2 + 2x_n}$. $x_1 = -2$

$\Rightarrow \quad x_2 = -2 - \frac{-2}{8} = -1.75 \quad \Rightarrow$

$x_3 = -1.75 - \dfrac{f(-1.75)}{f'(-1.75)} = -1.6978$

**4.** $f(x) = x^5 - 10 \quad \Rightarrow \quad f'(x) = 5x^4$, so

$x_{n+1} = x_n - \frac{x_n^5 - 10}{5x_n^4}$. $x_1 = 1.5$

$\Rightarrow \quad x_2 = 1.5 - \dfrac{(1.5)^5 - 10}{5(1.5)^4} \approx 1.5951 \quad \Rightarrow$

$x_3 = 1.5951 - \dfrac{f(1.5951)}{f'(1.5951)} \approx 1.5850$.

**5.** Finding $\sqrt[4]{22}$ is equivalent to finding the positive root of $x^4 - 22 = 0$ so we take $f(x) = x^4 - 22 \quad \Rightarrow$ $f'(x) = 4x^3$ and $x_{n+1} = x_n - \frac{x_n^4 - 22}{4x_n^3}$. Taking $x_1 = 2$, we get $x_2 = 2.1875$, $x_3 \approx 2.16605940$, $x_4 \approx 2.16573684$ and $x_5 \approx x_6 \approx 2.16573677$. Thus $\sqrt[4]{22} \approx 2.16573677$ to eight decimal places.

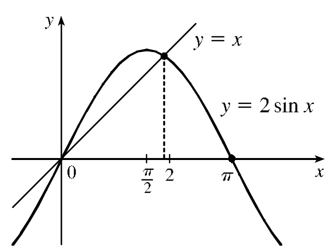
**6.** Finding $\sqrt[10]{100}$ is equivalent to finding the positive root of $x^{10} - 100 = 0$, so we take $f(x) = x^{10} - 100 \quad \Rightarrow$ $f'(x) = 10x^9$ and $x_{n+1} = x_n - \frac{x_n^{10} - 100}{10x_n^9}$. Taking $x_1 = 1.5$, we get $x_2 \approx 1.61012295$, $x_3 \approx 1.58659987$, $x_4 \approx 1.58490143$, $x_5 \approx 1.58489319$, and $x_6 \approx 1.58489319$. Thus $\sqrt[10]{100} \approx 1.58489319$ to eight decimal places.

**7.** $f(x) = x^3 - 2x - 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 2$, so $x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 1}{3x_n^2 - 2}$. Taking $x_1 = 1.5$, we get $x_2 \approx 1.631579$, $x_3 \approx 1.618184$, $x_4 \approx 1.618034$ and $x_5 \approx 1.618034$. So the root is 1.618034 to six decimal places.

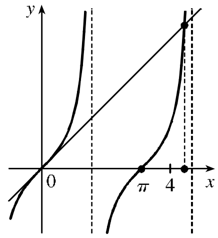
**8.** $f(x) = x^3 + x^2 + x - 2 \quad \Rightarrow \quad f'(x) = 3x^2 + 2x + 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 + x_n^2 + x_n - 2}{3x_n^2 + 2x_n - 1}$. Taking $x_1 = 1$, we get $x_2 \approx 0.833333$, $x_3 \approx 0.810916$, $x_4 \approx 0.810536$, and $x_5 \approx 0.810536$. So the root is 0.810536 to six decimal places.

**9.** $f(x) = x^4 + x^3 - 22x^2 - 2x + 41 \quad \Rightarrow$ $f'(x) = 4x^3 + 3x^2 - 44x - 2$, so $x_{n+1} = x_n - \dfrac{x_n^4 + x_n^3 - 22x_n^2 + 41}{4x_n^3 + 3x_n^2 - 44x_n - 2}$. Taking $x_1 = 4$, we get $x_2 \approx 3.992063$, $x_3 = 3.992020$, and $x_4 \approx 3.992020$. So the root in the interval $[3, 4]$ is 3.992020 to six decimal places.

**10.**




From the graph it appears that there is a root near 2, so we take $x_1 = 2$. Write the equation as $f(x) = 2\sin x - x = 0$. Then $f'(x) = 2\cos x - 1$, so $x_{n+1} = x_n - \dfrac{2\sin x_n - x_n}{2\cos x_n - 1}$ $\Rightarrow \quad x_1 = 2$, $x_2 \approx 1.900996$, $x_3 \approx 1.895512$, $x_4 \approx 1.895494 \approx x_5$. So the root is 1.895494, to six decimal places.

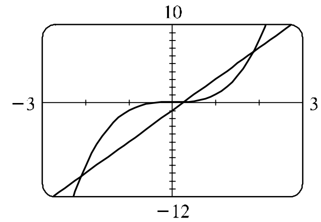
**11.** 


From the graph, it appears there is a root near 4.5. So we take $x_1 = 4.5$. Write the equation as $f(x) = \tan x - x = 0$. Then $f'(x) = \sec^2 x - 1$, so $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\sec^2 x_n - 1}$.
$x_1 = 4.5$, $x_2 \approx 4.493614$, $x_3 \approx 4.493410$,
$x_4 \approx 4.493409 \approx x_5$. To six decimal places, the root is 4.493409.

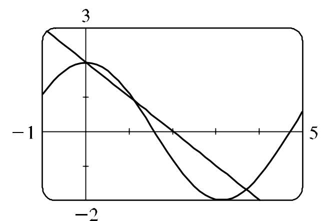
**12.** $f(x) = x^5 - 5x + 2 \;\Rightarrow\; f'(x) = 5x^4 - 5$, so
$x_{n+1} = x_n - \dfrac{x_n^5 - 5x_n + 2}{5x_n^4 - 5}$. Observe that $f(-2) = -20$,
$f(-1) = 6$, $f(0) = 2$, $f(1) = -2$ and $f(2) = 24$ so there are roots in $[-2,-1]$, $[0,1]$ and $[1,2]$. A sketch shows that these are the only intervals with roots.
$[-2,-1]$: $x_1 = -1.5$, $x_2 \approx -1.593846$, $x_3 \approx -1.582241$,
$x_4 \approx -1.582036$, $x_4 \approx -1.582036$
$[0,1]$: $x_1 = 0.5$, $x_2 = 0.4$, $x_3 \approx 0.402102$, $x_4 \approx 0.402102$
$[1,2]$: $x_1 = 1.5$, $x_3 \approx 1.396923$, $x_3 \approx 1.373078$,
$x_4 \approx 1.371885$, $x_5 \approx 1.371882$, $x_6 \approx 1.371882$
To six decimal places, the roots are $-1.582036$, 0.402102 and 1.371882.

**13.** $f(x) = x^4 + x^2 - x - 1 \;\Rightarrow\; f'(x) = 4x^3 + 2x - 1$, so
$x_{n+1} = x_n - \dfrac{x_n^4 + x_n^2 - x_n - 1}{4x_n^3 + 2x_n - 1}$. Note that $f(1) = 0$, so $x = 1$ is a root. Also $f(-1) = 2$ and $f(0) = -1$, so there is a root in $[-1, 0]$. A sketch shows that these are the only roots. Taking $x_1 = -0.5$, we have $x_2 = -0.575$,
$x_3 \approx -0.569867$, $x_4 \approx -0.569840$ and $x_5 \approx -0.569840$.
The roots are 1 and $-0.569840$, to six decimal places.

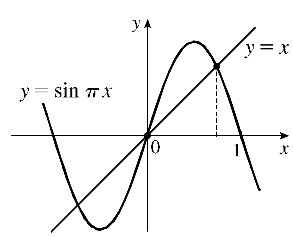
**14.** $f(x) = (x-2)^4 - \frac{1}{2}x \;\Rightarrow\; f'(x) = 4(x-2)^3 - \frac{1}{2}$, so
$x_{n+1} = x_n - \dfrac{(x_n - 2)^4 - \frac{1}{2}x_n}{4(x_n - 2)^3 - \frac{1}{2}}$. Observe that $f(1) = \frac{1}{2}$,
$f(2) = -1$, $f(3) = -\frac{1}{2}$ and $f(4) = 14$ so there are roots in $[1,2]$ and $[3,4]$ and a sketch shows that these are the only roots. Taking $x_1 = 1$, we get $x_2 \approx 1.111111$,
$x_3 \approx 1.131883$, $x_4 \approx 1.132529$ and $x_5 \approx 1.132529$.
Taking $x_1 = 3$, we get $x_2 \approx 3.142857$, $x_3 \approx 3.118267$,
$x_4 \approx 3.117350$, $x_5 \approx 3.117349$ and $x_6 \approx 3.117349$. To six decimal places, the roots are 1.132529 and 3.117349.

**15.** 


From the graph, we see that $y = x^3$ and $y = 4x - 1$ intersect three times. Good first approximations are $x = -2$, $x = 0$, and $x = 2$. $f(x) = x^3 - 4x + 1 \;\Rightarrow\; f'(x) = 3x^2 - 4$,
so $x_{n+1} = x_n - \dfrac{x_n^3 - 4x_n + 1}{3x_n^2 - 4}$.
$x_1 = -2$, $x_2 = -2.125$,
$x_3 \approx -2.114975$, $x_4 \approx -2.114908 \approx x_5$
$x_1 = 0$, $x_2 = 0.25$, $x_3 \approx 0.254098$, $x_4 \approx 0.254102 \approx x_5$
$x_1 = 2$, $x_2 = 1.875$, $x_3 \approx 1.860979$, $x_4 \approx 1.860806 \approx x_5$
To six decimal places, the roots are $-2.114908$, 0.254102, and 1.860806.

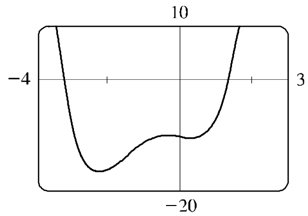
**16.** 


From the graph and by inspection, $x = 0$ is a root. Also, $y = 2\cos x$ and $y = 2 - x$ intersect at $x \approx 1$ and at $x \approx 3.5$.
$f(x) = 2\cos x + x - 2 \;\Rightarrow\; f'(x) = -2\sin x + 1$, so
$x_{n+1} = x_n - \dfrac{2\cos x_n + x_n - 2}{-2\sin x_n + 1}$.
$x_1 = 1$, $x_2 \approx 1.118026$, $x_3 \approx 1.109188$,
$x_4 \approx 1.109144 \approx x_5$
$x_1 = 3.5$, $x_2 \approx 3.719159$, $x_3 \approx 3.698331$,
$x_4 \approx 3.698154 \approx x_5$
To six decimal places, the roots are 0, 1.109144, and 3.698154.

**17.** 

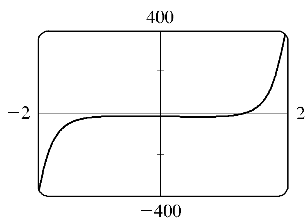

Clearly $x = 0$ is a root. From the sketch, there appear to be roots near $-0.75$ and 0.75. Write the equation as $f(x) = \sin \pi x - x = 0$. Then $f'(x) = \pi\cos \pi x - 1$, so
$x_{n+1} = x_n - \dfrac{\sin \pi x_n - x_n}{\pi \cos \pi x_n - 1}$. Taking $x_1 = 0.75$ we get
$x_2 \approx 0.736685$, $x_3 \approx 0.736484 \approx x_4$. To six decimal places, the roots are 0, 0.736484 and $-0.736484$.

**18.**



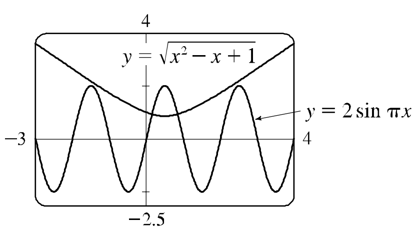

From the graph, there appear to be roots near $-3.2$ and $1.4$. Let $f(x) = x^4 + 3x^3 - x - 10$ $\Rightarrow$ $f'(x) = 4x^3 + 9x^2 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^4 + 3x_n^3 - x_n - 10}{4x_n^3 + 9x_n^2 - 1}$. Taking $x_1 = -3.2$, we get $x_2 \approx -3.20617358$, $x_3 \approx -3.20614267 \approx x_4$. Taking $x_1 = 1.4$, we get $x_2 \approx 1.37560834$, $x_3 \approx 1.37506496$, $x_4 \approx 1.37506470 \approx x_5$. To eight decimal places, the roots are $-3.20614267$ and $1.37506470$.

**19.**




From the graph, we see that the only root appears to be near 1.25. Let $f(x) = x^9 - x^6 + 2x^4 + 5x - 14$. Then $f'(x) = 9x^8 - 6x^5 + 8x^3 + 5$, so $x_{n+1} = x_n - \dfrac{x_n^9 - x_n^6 + 2x_n^4 + 5x_n - 14}{9x_n^8 - 6x_n^5 + 8x_n^3 + 5}$. Taking $x_1 = 1.25$, we get $x_2 \approx 1.23626314$, $x_3 \approx 1.23571823$, $x_4 \approx 1.23571742 \approx x_5$. To eight decimal places, the root of the equation is $1.23571742$.

**20.**

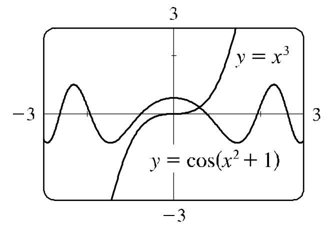



From the graph, we see that there are roots of this equation near 0.2 and 0.8. $f(x) = \sqrt{x^2 - x + 1} - 2\sin \pi x$ $\Rightarrow$ $f'(x) = \dfrac{2x - 1}{2\sqrt{x^2 - x + 1}} - 2\pi \cos \pi x$, so

$$x_{n+1} = x_n - \frac{\sqrt{x_n^2 - x_n + 1} - 2\sin \pi x_n}{\dfrac{2x_n - 1}{2\sqrt{x_n^2 - x_n + 1}} - 2\pi \cos \pi x_n}.$$

Taking $x_1 = 0.2$, we get $x_2 \approx 0.15212015$, $x_3 \approx 0.15438067$, $x_4 \approx 0.15438500 \approx x_5$. Taking $x_1 = 0.8$, we get $x_2 \approx 0.84787985$, $x_3 \approx 0.84561933$, $x_4 \approx 0.84561500 \approx x_5$. So, to eight decimal places, the roots of the equation are $0.15438500$ and $0.84561500$.

**21.**




From the graph, we see that the only root of this equation is near 0.6. $f(x) = \cos(x^2 + 1) - x^3$ $\Rightarrow$ $f'(x) = -2x \sin(x^2 + 1) - 3x^2$, so $x_{n+1} = x_n + \dfrac{\cos(x_n^2 + 1) - x_n^3}{2x_n \sin(x_n^2 + 1) + 3x_n^2}$. Taking $x_1 = 0.6$, we get $x_2 \approx 0.59699955$, $x_3 \approx 0.59698777 \approx x_3$. To eight decimal places, the root of the equation is $0.59698777$.