5.1 AREAS AND DISTANCES

A Click here for answers.

S Click here for solutions.

1. (a) Estimate the area under the graph of $f(x) = x^3 + 2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.
(b) Repeat part (a) using left endpoints.
(c) Repeat part (a) using midpoints.
(d) From your sketches in parts (a), (b), and (c), which appears to be the best estimate?

2–4 ▪ Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

2. $f(x) = \sqrt[3]{x}, \quad 0 \leqslant x \leqslant 8$

3. $f(x) = 5 + \sqrt[3]{x}, \quad 1 \leqslant x \leqslant 8$

4. $f(x) = x + \ln x, \quad 2 \leqslant x \leqslant 6$

5. Determine a region whose area is equal to

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$$

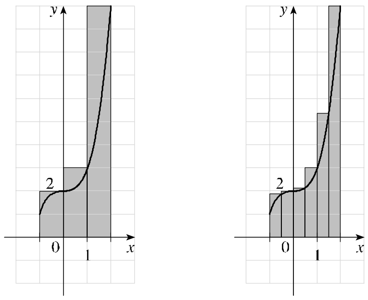
Do not evaluate the limit.

5.1 ANSWERS

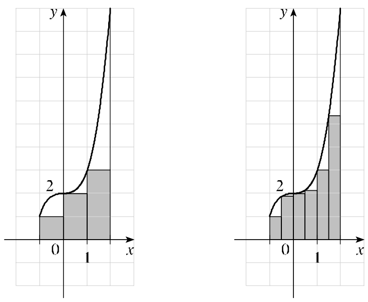
E Click here for exercises.

S Click here for solutions.

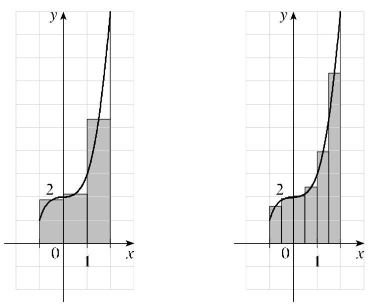
1. (a) 15, 12.1875



(b) 6, 7.6875



(c) 9.375, 9.65625



(d) M_6

2. $\displaystyle\lim_{n\to\infty} \frac{8}{n} \sum_{i=1}^{n} \sqrt[3]{\frac{8i}{n}}$

3. $\displaystyle\lim_{n\to\infty} \frac{7}{n} \sum_{i=1}^{n} \left(5 + \sqrt[3]{1 + \frac{7i}{n}}\right)$

4. $\displaystyle\lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[2 + \frac{4i}{n} + \ln\left(2 + \frac{4i}{n}\right)\right]$

5. The area of the region under the graph of $\sqrt{x}$ on the interval $[1, 4]$, or the area of the region lying under the graph of $\sqrt{x+1}$ on the interval $[0, 3]$

5.1 SOLUTIONS

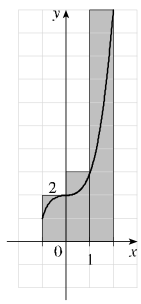
E Click here for exercises.

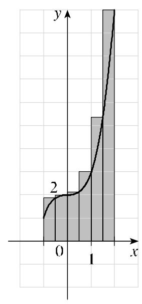
1. (a) $f(x) = x^3 + 2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \quad \Rightarrow$

$$R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2)$$
$$= 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 10 = 15$$

$$\Delta x = \frac{2-(-1)}{6} = 0.5 \quad \Rightarrow$$

$$R_6 = 0.5\,[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$$
$$= 0.5\,(1.875 + 2 + 2.125 + 3 + 5.375 + 10)$$
$$= 0.5\,(24.375) = 12.1875$$

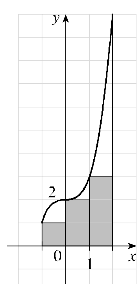


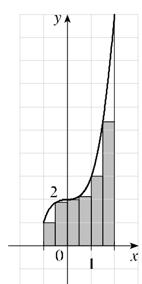


(b) $L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1)$

$$= 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 6$$

$$L_6 = 0.5\,[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$$
$$= 0.5\,(1 + 1.875 + 2 + 2.125 + 3 + 5.375)$$
$$= 0.5\,(15.375) = 7.6875$$

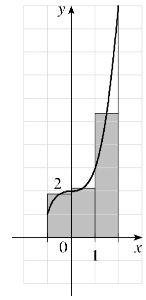


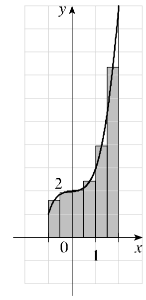


(c) $M_3 = 1 \cdot f(-0.5) + 1 \cdot f(0.5) + 1 \cdot f(1.5)$

$$= 1 \cdot 1.875 + 1 \cdot 2.125 + 1 \cdot 5.375 = 9.375$$

$$M_6 = 0.5\,[f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)]$$
$$= 0.5\,(1.578125 + 1.984375 + 2.015625 + 2.421875 + 3.953125 + 7.359375)$$
$$= 0.5\,(19.3125) = 9.65625$$





(d) M_6 appears to be the best estimate.

2. $f(x) = \sqrt[3]{x}, 0 \le x \le 8 \quad \Rightarrow \quad \Delta x = \dfrac{8-0}{n} = \dfrac{8}{n}$,

$x_i = 0 + i\,\Delta x = \dfrac{8i}{n} \quad \Rightarrow$

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$
$$= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \sqrt[3]{\frac{8i}{n}} \cdot \frac{8}{n}$$

Thus, $A = \displaystyle\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{8}{n} \sum_{i=1}^{n} \sqrt[3]{\frac{8i}{n}}$.

3. $f(x) = 5 + \sqrt[3]{x}, 1 \le x \le 8 \quad \Rightarrow \quad \Delta x = \dfrac{8-1}{n} = \dfrac{7}{n}$,

$x_i = 1 + i\,\Delta x = 1 + \dfrac{7i}{n} \quad \Rightarrow$

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$
$$= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(5 + \sqrt[3]{1 + \frac{7i}{n}}\right) \cdot \frac{7}{n}$$

Thus, $A = \displaystyle\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{7}{n} \sum_{i=1}^{n} \left(5 + \sqrt[3]{1 + \frac{7i}{n}}\right)$.

4. $f(x) = x + \ln x, 2 \le x \le 6 \quad \Rightarrow \quad \Delta x = \dfrac{6-2}{n} = \dfrac{4}{n}$,

$x_i = 2 + i\,\Delta x = 2 + \dfrac{4i}{n} \quad \Rightarrow$

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$
$$= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left[2 + \frac{4i}{n} + \ln\left(2 + \frac{4i}{n}\right)\right] \cdot \frac{4}{n}$$

Thus,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[2 + \frac{4i}{n} + \ln\left(2 + \frac{4i}{n}\right)\right].$$

5. The two most obvious ways to interpret $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{3}{n}\sqrt{1+\frac{3i}{n}}$ are: as the area of the region under the graph of $\sqrt{x}$ on the interval $[1,4]$, or as the area of the region lying under the graph of $\sqrt{x+1}$ on the interval $[0,3]$, since for $y=\sqrt{x+1}$ on $[0,3]$ with partition points $x_i=\frac{i}{3n}$, $\Delta x=\frac{1}{3n}$ and $x_i^*=x_i$, the expression for the area is $A=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}f\left(x_i^*\right)\Delta x=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\sqrt{1+\frac{3i}{n}}\left(\frac{3}{n}\right)$.