5.3 EVALUATING DEFINITE INTEGRALS

A Click here for answers.

S Click here for solutions.

1–2 ▪ Verify by differentiation that the formula is correct.

1. $\int \sin^2 x\,dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4} + C$

2. $\int x^2 \sin x\,dx = -x^2 \cos x + 2\int x \cos x\,dx$

3–4 ▪ Find the general indefinite integral.

3. $\int \sqrt{x}(x^2 - 1/x)\,dx$

4. $\int (2x + \sec x \tan x)\,dx$

5–43 ▪ Evaluate the integral.

5. $\int_0^1 (1 - 2x - 3x^2)\,dx$

6. $\int_1^2 (5x^2 - 4x + 3)\,dx$

7. $\int_0^1 (y^9 - 2y^5 + 3y)\,dy$

8. $\int_1^3 \left(\dfrac{1}{t^2} - \dfrac{1}{t^4}\right) dt$

9. $\int_{-2}^4 (3x - 5)\,dx$

10. $\int_1^2 x^{-2}\,dx$

11. $\int_0^4 \sqrt{x}\,dx$

12. $\int_0^1 x^{3/7}\,dx$

13. $\int_{-1}^4 \pi\,dx$

14. $\int_{-4}^{-1} \sqrt{3}\,dx$

15. $\int_3^6 (4 - 7x)\,dx$

16. $\int_1^4 (2x^2 - 3x + 1)\,dx$

17. $\int_1^3 (x - 2)(x + 3)\,dx$

18. $\int_0^{\pi/3} (1 - 2\cos x)\,dx$

19. $\int_0^1 (5\cos x + 4x)\,dx$

20. $\int_3^3 \sqrt{x^5 + 2}\,dx$

21. $\int_{\pi/4}^{\pi/3} \sin t\,dt$

22. $\int_1^2 \left(x + \dfrac{1}{x}\right)^2 dx$

23. $\int_0^1 \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right) dx$

24. $\int_1^8 \dfrac{x - 1}{\sqrt[3]{x^2}}\,dx$

25. $\int_{\ln 3}^{\ln 6} 8e^x\,dx$

26. $\int_8^9 2^t\,dt$

27. $\int_{-e^2}^{-e} \dfrac{3}{x}\,dx$

28. $\int_1^{\sqrt{3}} \dfrac{6}{1 + x^2}\,dx$

29. $\int_0^{0.5} \dfrac{dx}{\sqrt{1 - x^2}}$

30. $\int_1^2 \dfrac{t^6 - t^2}{t^4}\,dt$

31. $\int_1^2 \dfrac{x^2 + 1}{\sqrt{x}}\,dx$

32. $\int_0^2 (x^3 - 1)^2\,dx$

33. $\int_0^1 u(\sqrt{u} + \sqrt[3]{u})\,du$

34. $\int_{-1}^2 |x - x^2|\,dx$

35. $\int_{-2}^3 |x^2 - 1|\,dx$

36. $\int_1^{-1} (x - 1)(3x + 2)\,dx$

37. $\int_1^4 \left(\sqrt{t} - \dfrac{2}{\sqrt{t}}\right) dt$

38. $\int_1^8 \left(\sqrt[3]{r} + \dfrac{1}{\sqrt[3]{r}}\right) dr$

39. $\int_{-1}^0 (x + 1)^3\,dx$

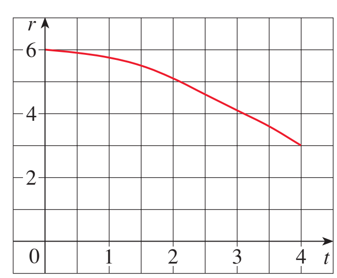
40. $\int_{-5}^{-2} \dfrac{x^4 - 1}{x^2 + 1}\,dx$

41. $\int_{\pi/6}^{\pi/3} \csc^2\theta\,d\theta$

42. $\int_{\pi/3}^{\pi/2} \csc x \cot x\,dx$

43. $\int_0^2 (x^2 - |x - 1|)\,dx$

44. Water leaked from a tank at a rate of $r(t)$ liters per hour, where the graph of r is as shown. Express the total amount of water that leaked out during the first four hours as a definite integral. Then use the Midpoint Rule to estimate that amount.



5.3 ANSWERS

E Click here for exercises.

S Click here for solutions.

3. $\frac{2}{7}x^{7/2} - 2x^{1/2} + C$

4. $x^2 + \sec x + C$

5. -1

6. $\frac{26}{3}$

7. $\frac{19}{15}$

8. $\frac{28}{81}$

9. -12

10. $\frac{1}{2}$

11. $\frac{16}{3}$

12. $\frac{7}{10}$

13. 5π

14. $3\sqrt{3}$

15. $-\frac{165}{2}$

16. $\frac{45}{2}$

17. $\frac{2}{3}$

18. $\frac{\pi}{3} - \sqrt{3}$

19. $5\sin 1 + 2$

20. 0

21. $\frac{\sqrt{2}-1}{2}$

22. $\frac{29}{6}$

23. 1

24. $\frac{33}{4}$

25. 24

26. $\dfrac{2^8}{\ln 2}$

27. -3

28. $\frac{\pi}{2}$

29. $\frac{\pi}{6}$

30. $\frac{11}{6}$

31. $\frac{6}{5}\left(3\sqrt{2} - 2\right)$

32. $\frac{86}{7}$

33. $\frac{29}{35}$

34. $\frac{11}{6}$

35. $\frac{28}{3}$

36. 2

37. $\frac{2}{3}$

38. $\frac{63}{4}$

39. $\frac{1}{4}$

40. 36

41. $\frac{2}{3}\sqrt{3}$

42. $-1 + \frac{2}{3}\sqrt{3}$

43. $\frac{5}{3}$

44. $\int_0^4 r(t)\,dt \approx 19.6\,\mathrm{L}$

5.3 SOLUTIONS

E Click here for exercises.

1. $\dfrac{d}{dx}\left(\dfrac{x}{2}-\dfrac{\sin 2x}{4}+C\right)=\frac{1}{2}-\frac{1}{4}(\cos 2x)(2)+0$
$=\frac{1}{2}-\frac{1}{2}\cos 2x=\frac{1}{2}-\frac{1}{2}\left(1-2\sin^2 x\right)=\sin^2 x$

2. $\dfrac{d}{dx}\left(-x^2\cos x+2\int x\cos x\,dx\right)$
$=-x^2(-\sin x)-2x\cos x+2x\cos x=x^2\sin x$

3. $\int\sqrt{x}\left(x^2-1/x\right)dx=\int\left(x^{5/2}-x^{-1/2}\right)dx$
$=\frac{2}{7}x^{7/2}-2x^{1/2}+C$

4. $\int(2x+\sec x\tan x)\,dx=x^2+\sec x+C$

5. $\int_0^1\left(1-2x-3x^2\right)dx=\left[x-2\cdot\frac{1}{2}x^2-3\cdot\frac{1}{3}x^3\right]_0^1$
$=\left[x-x^2-x^3\right]_0^1=(1-1-1)-0=-1$

6. $\int_1^2\left(5x^2-4x+3\right)dx=\left[5\cdot\frac{1}{3}x^3-4\cdot\frac{1}{2}x^2+3x\right]_1^2$
$=5\cdot\frac{8}{3}-4\cdot 2+6-\left(\frac{5}{3}-2+3\right)=\frac{26}{3}$

7. $\int_0^1\left(y^9-2y^5+3y\right)dy=\left[\frac{1}{10}y^{10}-2\left(\frac{1}{6}y^6\right)+3\left(\frac{1}{2}y^2\right)\right]_0^1$
$=\left(\frac{1}{10}-\frac{1}{3}+\frac{3}{2}\right)-0=\frac{19}{15}$

8. $\displaystyle\int_1^3\left(\frac{1}{t^2}-\frac{1}{t^4}\right)dt=\int_1^3\left(t^{-2}-t^{-4}\right)dt$
$\displaystyle=\left[\frac{t^{-1}}{-1}-\frac{t^{-3}}{-3}\right]_1^3=\left[\frac{1}{3t^3}-\frac{1}{t}\right]_1^3$
$\displaystyle=\left(\frac{1}{81}-\frac{1}{3}\right)-\left(\frac{1}{3}-1\right)=\frac{28}{81}$

9. $\int_{-2}^4(3x-5)\,dx=\left(3\cdot\frac{1}{2}x^2-5x\right)\Big|_{-2}^4$
$=(3\cdot 8-5\cdot 4)-[3\cdot 2-(-10)]=-12$

10. $\int_1^2 x^{-2}\,dx=\left[-x^{-1}\right]_1^2=[-1/x]_1^2=-\frac{1}{2}+1=\frac{1}{2}$

11. $\displaystyle\int_0^4\sqrt{x}\,dx=\int_0^4 x^{1/2}\,dx=\left[\frac{x^{3/2}}{3/2}\right]_0^4=\left[\frac{2x^{3/2}}{3}\right]_0^4$
$\displaystyle=\frac{2(4)^{3/2}}{3}-0=\frac{16}{3}$

12. $\displaystyle\int_0^1 x^{3/7}\,dx=\left[\frac{x^{10/7}}{10/7}\right]_0^1=\left[\tfrac{7}{10}x^{10/7}\right]_0^1=\frac{7}{10}-0=\frac{7}{10}$

13. $\int_{-1}^4\pi\,dx=\pi[4-(-1)]=5\pi$

14. $\int_{-4}^{-1}\sqrt{3}\,dx=\sqrt{3}(-1+4)=3\sqrt{3}$

15. $\int_3^6(4-7x)\,dx=\int_3^6 4\,dx-\int_3^6 7x\,dx$
$=4(6-3)-7\int_3^6 x\,dx$
$=12-7\cdot\frac{1}{2}\left(6^2-3^2\right)$
$=12-\frac{7}{2}(27)=-\frac{165}{2}$

16. $\int_1^4\left(2x^2-3x+1\right)dx=2\int_1^4 x^2\,dx-3\int_1^4 x\,dx+\int_1^4 1\,dx$
$=2\cdot\frac{1}{3}\left(4^3-1^3\right)-3\cdot\frac{1}{2}\left(4^2-1^2\right)+1(4-1)$
$=\frac{45}{2}=22.5$

17. $\int_1^3(x-2)(x+3)\,dx=\int_1^3\left(x^2+x-6\right)dx$
$=\int_1^3 x^2\,dx+\int_1^3 x\,dx+\int_1^3(-6)\,dx$
$=\frac{1}{3}\left(3^3-1^3\right)+\frac{1}{2}\left(3^2-1^2\right)+(-6)(3-1)=\frac{2}{3}$

18. $\int_0^{\pi/3}(1-2\cos x)\,dx=\int_0^{\pi/3}1\,dx-2\int_0^{\pi/3}\cos x\,dx$
$=1\left(\frac{\pi}{3}-0\right)-2\sin\frac{\pi}{3}=\frac{\pi}{3}-\sqrt{3}$

19. $\int_0^1(5\cos x+4x)\,dx=5\int_0^1\cos x\,dx+4\int_0^1 x\,dx$
$=5\sin 1+4\cdot\frac{1}{2}\left(1^2-0^2\right)=5\sin 1+2$

20. $\int_3^3\sqrt{x^5+2}\,dx=0$ since the lower and upper limits are equal.

21. $\int_{\pi/4}^{\pi/3}\sin t\,dt=[-\cos t]_{\pi/4}^{\pi/3}$
$=-\cos\frac{\pi}{3}+\cos\frac{\pi}{4}=-\frac{1}{2}+\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-1}{2}$

22. $\displaystyle\int_1^2\left(x+\frac{1}{x}\right)^2dx=\int_1^2\left(x^2+2+x^{-2}\right)dx$
$\displaystyle=\left[\frac{x^3}{3}+2x+\frac{x^{-1}}{-1}\right]_1^2=\left[\frac{x^3}{3}+2x-\frac{1}{x}\right]_1^2$
$=\left(\frac{8}{3}+4-\frac{1}{2}\right)-\left(\frac{1}{3}+2-1\right)=\frac{29}{6}$

23. $\int_0^1\left[\sqrt[4]{x^5}+\sqrt[5]{x^4}\right]dx=\int_0^1\left(x^{5/4}+x^{4/5}\right)dx$
$\displaystyle=\left[\frac{x^{9/4}}{9/4}+\frac{x^{9/5}}{9/5}\right]_0^1=\left[\tfrac{4}{9}x^{9/4}+\tfrac{5}{9}x^{9/5}\right]_0^1$
$=\frac{4}{9}+\frac{5}{9}-0=1$

24. $\displaystyle\int_1^8\frac{x-1}{\sqrt[3]{x^2}}dx=\int_1^8\left(x^{1/3}-x^{-2/3}\right)dx$
$\displaystyle=\left[\frac{x^{4/3}}{4/3}-\frac{x^{1/3}}{1/3}\right]_1^8$
$=\left[\frac{3}{4}x^{4/3}-3x^{1/3}\right]_1^8$
$=\left(\frac{3}{4}\cdot 16-3\cdot 2\right)-\left(\frac{3}{4}-3\right)=\frac{33}{4}$

25. $\int_{\ln 3}^{\ln 6}8e^x\,dx=[8e^x]_{\ln 3}^{\ln 6}=8\left(e^{\ln 6}-e^{\ln 3}\right)$
$=8(6-3)=24$

26. $\displaystyle\int_8^9 2^t\,dt=\left[\frac{1}{\ln 2}2^t\right]_8^9=\frac{1}{\ln 2}\left(2^9-2^8\right)=\frac{2^8}{\ln 2}$

27. $\int_{-e^2}^{-e}(3/x)\,dx=[3\ln|x|]_{-e^2}^{-e}=3\ln e-3\ln\left(e^2\right)$
$=3\cdot 1-3\cdot 2=-3$

28. $\displaystyle\int_1^{\sqrt{3}} \frac{6}{1+x^2}\,dx = 6\left[\tan^{-1} x\right]_1^{\sqrt{3}}$
$= 6\tan^{-1}\sqrt{3} - 6\tan^{-1} 1$
$= 6\cdot\frac{\pi}{3} - 6\cdot\frac{\pi}{4} = \frac{\pi}{2}$

29. $\displaystyle\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^{0.5} = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}$

30. $\displaystyle\int_1^2 \frac{t^6-t^2}{t^4}\,dt = \int_1^2 \left(t^2 - t^{-2}\right)dt = \left[\frac{t^3}{3} - \frac{t^{-1}}{-1}\right]_1^2$
$= \displaystyle\left[\frac{t^3}{3} + \frac{1}{t}\right]_1^2 = \left(\frac{8}{3}+\frac{1}{2}\right) - \left(\frac{1}{3}+1\right) = \frac{11}{6}$

31. $\displaystyle\int_1^2 \frac{x^2+1}{\sqrt{x}}\,dx = \int_1^2 \left(x^{3/2} + x^{-1/2}\right)dx$
$= \displaystyle\left[\frac{x^{5/2}}{5/2} + \frac{x^{1/2}}{1/2}\right]_1^2 = \left[\tfrac{2}{5}x^{5/2} + 2x^{1/2}\right]_1^2$
$= \left(\tfrac{2}{5}4\sqrt{2} + 2\sqrt{2}\right) - \left(\tfrac{2}{5}+2\right)$
$= \frac{18\sqrt{2}-12}{5} = \frac{6}{5}\left(3\sqrt{2}-2\right)$

32. $\int_0^2 \left(x^3-1\right)^2 dx = \int_0^2 \left(x^6 - 2x^3 + 1\right)dx$
$= \left[\frac{1}{7}x^7 - 2\left(\frac{1}{4}x^4\right) + x\right]_0^2 = \left(\frac{128}{7} - 2\cdot 4 + 2\right) - 0 = \frac{86}{7}$

33. $\int_0^1 u\left(\sqrt{u} + \sqrt[3]{u}\right)du$
$= \displaystyle\int_0^1 \left(u^{3/2} + u^{4/3}\right)du = \left[\frac{u^{5/2}}{5/2} + \frac{u^{7/3}}{7/3}\right]_0^1$
$= \left[\frac{2}{5}u^{5/2} + \frac{3}{7}u^{7/3}\right]_0^1 = \frac{2}{5} + \frac{3}{7} = \frac{29}{35}$

34. $\int_{-1}^2 \left|x - x^2\right| dx$
$= \int_{-1}^0 \left(x^2 - x\right)dx + \int_0^1 \left(x - x^2\right)dx + \int_1^2 \left(x^2 - x\right)dx$
$= \displaystyle\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$
$= 0 - \left(-\frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - 0 + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$
$= \frac{11}{6}$

35. $\int_{-2}^3 \left|x^2 - 1\right| dx$
$= \int_{-2}^{-1} \left(x^2-1\right)dx + \int_{-1}^1 \left(1-x^2\right)dx + \int_1^3 \left(x^2-1\right)dx$
$= \displaystyle\left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^1 + \left[\frac{x^3}{3} - x\right]_1^3$
$= \left(-\frac{1}{3}+1\right) - \left(-\frac{8}{3}+2\right) + \left(1-\frac{1}{3}\right)$
$- \left(-1+\frac{1}{3}\right) + (9-3) - \left(\frac{1}{3}-1\right)$
$= \frac{28}{3}$

36. $\int_1^{-1} (x-1)(3x+2)\,dx = -\int_{-1}^1 \left(3x^2 - x - 2\right)dx$
$= \displaystyle -\left[3\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^1$
$= \displaystyle\left[-x^3 + \frac{x^2}{2} + 2x\right]_{-1}^1$
$= \left(-1+\frac{1}{2}+2\right) - \left(1+\frac{1}{2}-2\right) = 2$

37. $\displaystyle\int_1^4 \left(\sqrt{t} - \frac{2}{\sqrt{t}}\right)dt = \int_1^4 \left(t^{1/2} - 2t^{-1/2}\right)dt$
$= \displaystyle\left[\frac{t^{3/2}}{3/2} - 2\frac{t^{1/2}}{1/2}\right]_1^4 = \left[\tfrac{2}{3}t^{3/2} - 4t^{1/2}\right]_1^4$
$= \left(\frac{2}{3}\cdot 8 - 4\cdot 2\right) - \left(\frac{2}{3} - 4\right) = \frac{2}{3}$

38. $\displaystyle\int_1^8 \left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}}\right)dr = \int_1^8 \left(r^{1/3} + r^{-1/3}\right)dr$
$= \displaystyle\left[\frac{r^{4/3}}{4/3} + \frac{r^{2/3}}{2/3}\right]_1^8 = \left[\tfrac{3}{4}r^{4/3} + \tfrac{3}{2}r^{2/3}\right]_1^8$
$= \left(\frac{3}{4}\cdot 16 + \frac{3}{2}\cdot 4\right) - \left(\frac{3}{4}+\frac{3}{2}\right) = \frac{63}{4}$

39. $\int_{-1}^0 (x+1)^3\,dx = \int_{-1}^0 \left(x^3 + 3x^2 + 3x + 1\right)dx$
$= \displaystyle\left[\frac{x^4}{4} + 3\frac{x^3}{3} + 3\frac{x^2}{2} + x\right]_{-1}^0$
$= 0 - \left[\frac{1}{4} - 1 + \frac{3}{2} - 1\right] = 2 - \frac{7}{4} = \frac{1}{4}$

40. $\displaystyle\int_{-5}^{-2} \frac{x^4-1}{x^2+1}\,dx = \int_{-5}^{-2} \left(x^2-1\right)dx$
$= \displaystyle\left[\frac{x^3}{3} - x\right]_{-5}^{-2} = \left(-\frac{8}{3}+2\right) - \left(\frac{-125}{3}+5\right) = 36$

41. $\int_{\pi/6}^{\pi/3} \csc^2\theta\,d\theta = \left[-\cot\theta\right]_{\pi/6}^{\pi/3} = -\cot\frac{\pi}{3} + \cot\frac{\pi}{6}$
$= -\frac{1}{3}\sqrt{3} + \sqrt{3} = \frac{2}{3}\sqrt{3}$

42. $\int_{\pi/3}^{\pi/2} \csc x\cot x\,dx = \left[-\csc x\right]_{\pi/3}^{\pi/2}$
$= -\csc\frac{\pi}{2} + \csc\frac{\pi}{3} = -1 + \frac{2}{3}\sqrt{3}$

43. $\int_0^2 \left(x^2 - |x-1|\right)dx$
$= \int_0^1 \left(x^2 + x - 1\right)dx + \int_1^2 \left(x^2 - x + 1\right)dx$
$= \displaystyle\left[\frac{x^3}{3} + \frac{x^2}{2} - x\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_1^2$
$= \left(\frac{1}{3}+\frac{1}{2}-1\right) - 0 + \left(\frac{8}{3}-2+2\right) - \left(\frac{1}{3}-\frac{1}{2}+1\right) = \frac{5}{3}$

44. Let w be the amount of water in the tank. We are given that the rate of water leaving the tank is $r(t) = -dw/dt$. So by the Net Change Theorem, the total loss of water from the tank after four hours is

$$\begin{aligned} w(0) - w(4) &= -\left[w(4) - w(0)\right] \\ &= -\int_0^4 w'(t)\,dt \\ &= \int_0^4 r(t)\,dt \end{aligned}$$

We use the Midpoint Rule with $n = 4$ and $\Delta t = 1$:

$$\begin{aligned} \int_0^4 r(t)\,dt &\approx \sum_{i=1}^4 r\left(\bar{t}_i\right)(1) \\ &= r(0.5) + r(1.5) + r(2.5) + r(3.5) \\ &\approx 5.9 + 5.4 + 4.7 + 3.6 \\ &= 19.6\text{ L} \end{aligned}$$