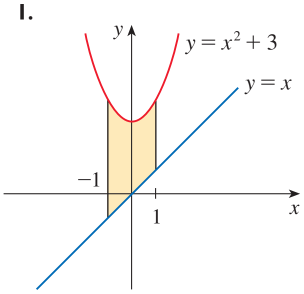
7.1 AREAS BETWEEN CURVES

A Click here for answers.

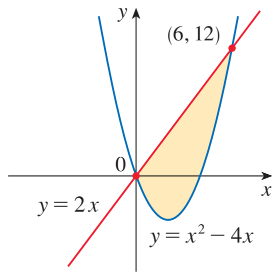
S Click here for solutions.

1–4 ▪ Find the area of the shaded region.

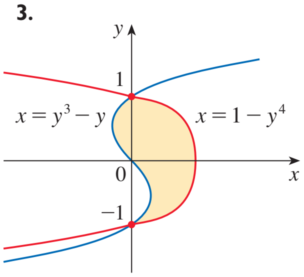
1.



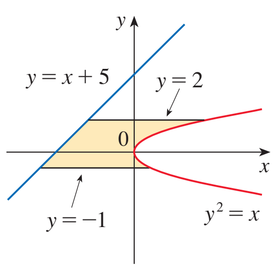
2.



3.



4.



5–10 ▪ Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

5. $y = 4x^2, \quad y = x^2 + 3$

6. $y = x + 1, \quad y = (x - 1)^2, \quad x = -1, \quad x = 2$

7. $y = x^2 + 1, \quad y = 3 - x^2, \quad x = -2, \quad x = 2$

8. $y^2 = x, \quad x - 2y = 3$

9. $y = 1/x, \quad x = 0, \quad y = 1, \quad y = 2$

10. $y = \cos x, \quad y = \sec^2 x, \quad x = -\pi/4, \quad x = \pi/4$

11–36 ▪ Sketch the region bounded by the given curves and find the area of the region.

11. $y = x, \quad y = x^3$

12. $y = \sqrt{x}, \quad y = x/2$

13. $y = \sqrt{x - 1}, \quad x - 3y + 1 = 0$

14. $y = x^4 - x^2, \quad y = 1 - x^2$

15. $y = x^2 + 2, \quad y = 2x + 5, \quad x = 0, \quad x = 6$

16. $x + y^2 = 2, \quad x + y = 0$

17. $y = x^2 + 3, \quad y = x, \quad x = -1, \quad x = 1$

18. $y = x^4, \quad y = -x - 1, \quad x = -2, \quad x = 0$

19. $y^2 = x, \quad y = x + 5, \quad y = -1, \quad y = 2$

20. $x + y^2 = 0, \quad x = y^2 + 1, \quad y = 0, \quad y = 3$

21. $y = x^2 - 4x, \quad y = 2x$

22. $x^2 + 2x + y = 0, \quad x + y + 2 = 0$

23. $y = 4 - x^2, \quad y = x + 2, \quad x = -3, \quad x = 0$

24. $y = x^2 + 2x + 2, \quad y = x + 4, \quad x = -3, \quad x = 2$

25. $y = x^3 - 4x^2 + 3x, \quad y = x^2 - x$

26. $y = x, \quad y = \sin x, \quad x = -\pi/4, \quad x = \pi/2$

27. $y = \sin x, \quad y = \cos 2x, \quad x = 0, \quad x = \pi/4$

28. $y = |x|, \quad y = (x + 1)^2 - 7, \quad x = -4$

29. $y = |x - 1|, \quad y = x^2 - 3, \quad x = 0$

30. $x = 3y, \quad x + y = 0, \quad 7x + 3y = 24$

31. $y = x\sqrt{1 - x^2}, \quad y = x - x^3$

32. $y = 1/x, \quad y = 1/x^2, \quad x = 1, \quad x = 2$

33. $y = x^2, \quad y = 2/(x^2 + 1)$

34. $y = 2^x, \quad y = 5^x, \quad x = -1, \quad x = 1$

35. $y = e^x, \quad y = e^{3x}, \quad x = 1$

36. $y = e^x, \quad y = e^{-x}, \quad x = -2, \quad x = 1$

37. Evaluate

$$\int_0^{\pi} \left| \sin x - \frac{2}{\pi} x \right| dx$$

and interpret it as the area of a region. Sketch the region.

38–39 ▪ Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the given curves.

38. $y = \sqrt{1 + x^3}, \quad y = 1 - x, \quad x = 2$

39. $y = x \tan x, \quad y = x$

40–41 ▪ Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the curves.

40. $y = 1 + 3x - 2x^2, \quad y = \sqrt{1 + x^4}$

41. $y = x^2 - x, \quad y = \sin(x^2)$

42–43 ■ Find the area of the region bounded by the given curves by two methods: (a) integrating with respect to x, and (b) integrating with respect to y.

42. $4x + y^2 = 0, \quad y = 2x + 4$

43. $x + 1 = 2(y - 2)^2, \quad x + 6y = 7$

44–45 ■ Use calculus to find the area of the triangle with the given vertices.

44. $(0, 0), (1, 8), (4, 3)$

45. $(-2, 5), (0, -3), (5, 2)$

46–48 ■ Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

46. $y = \sqrt{x + 1}, \quad y = x^2$

47. $y = x^4 - 1, \quad y = x \sin(x^2)$

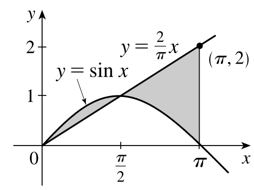
48. $y = x^2, \quad y = e^{-x^2}$

7.1 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{20}{3}$
2. 36
3. $\frac{8}{5}$
4. $\frac{33}{2}$
5. 4
6. $\frac{31}{6}$
7. 8
8. $\frac{32}{3}$
9. $\ln 2$
10. $2-\sqrt{2}$
11. $\frac{1}{2}$
12. $\frac{4}{3}$
13. $\frac{1}{6}$
14. $\frac{8}{5}$
15. 36
16. $\frac{9}{2}$
17. $\frac{20}{3}$
18. $\frac{32}{5}$
19. $\frac{33}{2}$
20. 21
21. 36
22. $\frac{9}{2}$
23. $\frac{31}{6}$
24. $\frac{49}{6}$
25. $\frac{71}{6}$
26. $\frac{5}{32}\pi^2+\frac{1}{\sqrt{2}}-2$
27. $\frac{1}{2}\left(3\sqrt{3}-\sqrt{2}-3\right)$
28. 34
29. $\frac{13}{3}$
30. 12
31. $\frac{1}{6}$
32. $\ln 2-\frac{1}{2}$
33. $\pi-\frac{2}{3}$
34. $\dfrac{16}{5\ln 5}-\dfrac{1}{2\ln 2}$
35. $\frac{1}{3}e^3-e+\frac{2}{3}$
36. $e^2+e+e^{-1}+e^{-2}-4$
37. $\frac{\pi}{2}$



38. 3.22
39. 0.13
40. 0.83
41. 0.81
42. 9
43. $\frac{1}{3}$
44. $\frac{29}{2}$
45. 25
46. 1.38
47. 1.78
48. 0.98

7.1 SOLUTIONS

E Click here for exercises.

1. $A = \int_{-1}^{1} \left[(x^2 + 3) - x \right] dx = 2 \int_0^1 (x^2 + 3)\, dx$

[by Theorem 5.5.7(a)]

$= 2 \left[\frac{1}{3}x^3 + 3x \right]_0^1 = 2 \left(\frac{1}{3} + 3 \right) = \frac{20}{3}$

2. $A = \int_0^6 \left[2x - (x^2 - 4x) \right] dx = \int_0^6 (6x - x^2)\, dx$

$= \left[3x^2 - \frac{1}{3}x^3 \right]_0^6 = 108 - 72 = 36$

3. $A = \int_{-1}^{1} \left[(1 - y^4) - (y^3 - y) \right] dy = 2 \int_0^1 (1 - y^4)\, dy$

[by Theorem 5.5.7(a)]

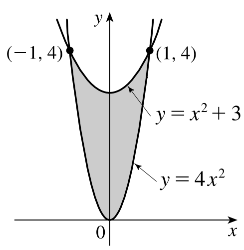
$= 2 \left[-\frac{1}{5}y^5 + y \right]_0^1 = 2 \left(-\frac{1}{5} + 1 \right) = \frac{8}{5}$

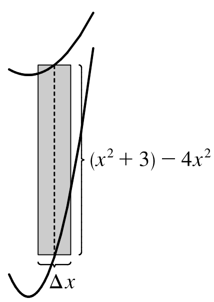
4. $A = \int_{-1}^{2} \left[y^2 - (y - 5) \right] dy = \left[\frac{1}{3}y^3 - \frac{1}{2}y^2 + 5y \right]_{-1}^{2}$

$= \left(\frac{8}{3} - 2 + 10 \right) - \left(-\frac{1}{3} - \frac{1}{2} - 5 \right) = \frac{33}{2}$

5. $A = \int_{-1}^{1} \left[(x^2 + 3) - 4x^2 \right] dx = 2 \int_0^1 (3 - 3x^2)\, dx$

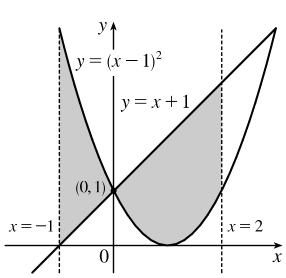
$= 2 \left[3x - x^3 \right]_0^1 = 2(3 - 1) = 4$

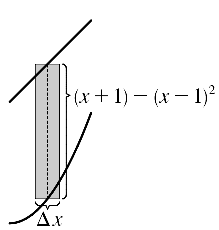




6. $x + 1 = (x - 1)^2 \;\Rightarrow\; x + 1 = x^2 - 2x + 1 \;\Rightarrow\; 0 = x^2 - 3x \;\Rightarrow\; 0 = x(x - 3) \;\Rightarrow\; x = 0$ or 3.

$$\begin{aligned} A &= \int_{-1}^{2} \left| (x + 1) - (x - 1)^2 \right| dx \\ &= \int_{-1}^{0} \left[(x - 1)^2 - (x + 1) \right] dx \\ &\quad + \int_0^2 \left[(x + 1) - (x - 1)^2 \right] dx \\ &= \int_{-1}^{0} (x^2 - 3x)\, dx + \int_0^2 (3x - x^2)\, dx \\ &= \left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 \right]_{-1}^{0} + \left[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 \right]_0^2 \\ &= 0 - \left(-\tfrac{1}{3} - \tfrac{3}{2} \right) + \left(6 - \tfrac{8}{3} \right) - 0 = \tfrac{31}{6} \end{aligned}$$





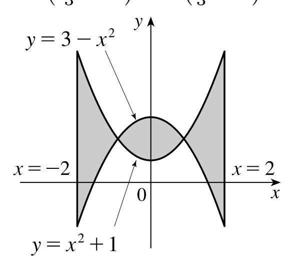
7. $A = \int_{-2}^{-1} \left[(x^2 + 1) - (3 - x^2) \right] dx + \int_{-1}^{1} \left[(3 - x^2) - (x^2 + 1) \right] dx + \int_1^2 \left[(x^2 + 1) - (3 - x^2) \right] dx$

$= \int_{-2}^{-1} (2x^2 - 2)\, dx + \int_{-1}^{1} (2 - 2x^2)\, dx + \int_1^2 (2x^2 - 2)\, dx$

$= 2 \int_0^1 (2 - 2x^2)\, dx + 2 \int_1^2 (2x^2 - 2)\, dx$ [symmetry]

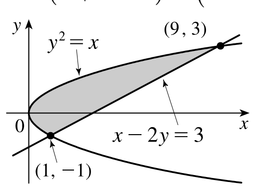
$= 2 \left[2x - \frac{2}{3}x^3 \right]_0^1 + 2 \left[\frac{2}{3}x^3 - 2x \right]_1^2$

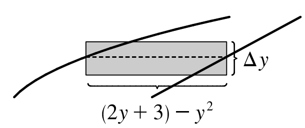
$= 2 \left(2 - \frac{2}{3} \right) + 2 \left(\frac{16}{3} - 4 \right) - 2 \left(\frac{2}{3} - 2 \right) = 8$



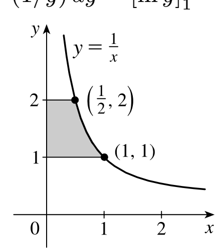
8. $A = \int_{-1}^{3} \left[(2y + 3) - y^2 \right] dy = \left[y^2 + 3y - \frac{1}{3}y^3 \right]_{-1}^{3}$

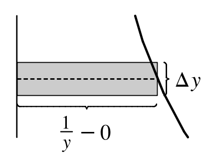
$= (9 + 9 - 9) - \left(1 - 3 + \frac{1}{3} \right) = \frac{32}{3}$





9. $A = \int_1^2 (1/y)\, dy = \left[\ln y \right]_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.69$

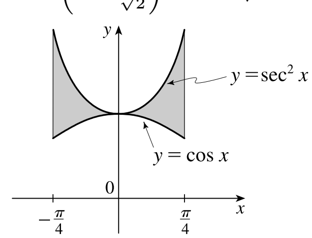


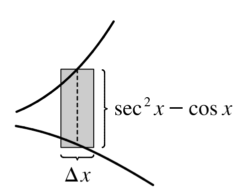


10. $A = \int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\, dx$

$= 2 \int_0^{\pi/4} (\sec^2 x - \cos x)\, dx = 2 \left[\tan x - \sin x \right]_0^{\pi/4}$

$= 2 \left(1 - \frac{1}{\sqrt{2}} \right) = 2 - \sqrt{2} \approx 0.59$

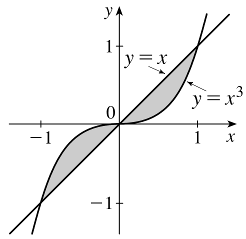




11. $A = \int_{-1}^{0} \left(x^3 - x\right) dx + \int_0^1 \left(x - x^3\right) dx$

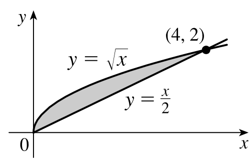
$= 2\int_0^1 \left(x - x^3\right) dx = 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$



12. $A = \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right]_0^4$

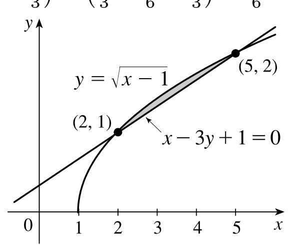
$= \left(\frac{16}{3} - 4\right) - 0 = \frac{4}{3}$



13. $A = \int_2^5 \left(\sqrt{x-1} - \frac{1}{3}x - \frac{1}{3}\right) dx$

$= \left[\frac{2}{3}(x-1)^{3/2} - \frac{1}{6}x^2 - \frac{1}{3}x\right]_2^5$

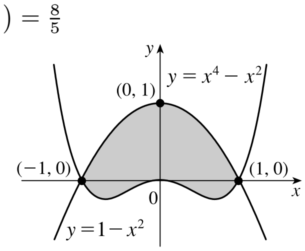
$= \left(\frac{16}{3} - \frac{25}{6} - \frac{5}{3}\right) - \left(\frac{2}{3} - \frac{4}{6} - \frac{2}{3}\right) = \frac{1}{6}$



14. $A = \int_{-1}^{1} \left[\left(1 - x^2\right) - \left(x^4 - x^2\right)\right] dx$

$= 2\int_0^1 \left(1 - x^4\right) dx = 2\left[x - \frac{1}{5}x^5\right]_0^1$

$= 2\left(1 - \frac{1}{5}\right) = \frac{8}{5}$



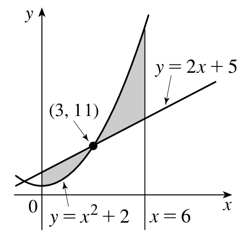
15. $A = \int_0^3 \left[(2x+5) - \left(x^2+2\right)\right] dx + \int_3^6 \left[\left(x^2+2\right) - (2x+5)\right] dx$

$= \int_0^3 \left(-x^2 + 2x + 3\right) dx + \int_3^6 \left(x^2 - 2x - 3\right) dx$

$= \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_0^3 + \left[\frac{1}{3}x^3 - x^2 - 3x\right]_3^6$

$= (-9 + 9 + 9) - 0 + (72 - 36 - 18) - (9 - 9 - 9)$

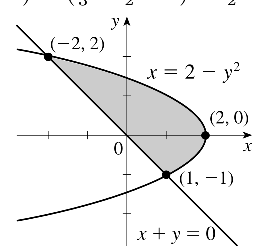
$= 36$



16. $A = \int_{-1}^{2} \left[2 - y^2 - (-y)\right] dy = \int_{-1}^{2} \left(-y^2 + y + 2\right) dy$

$= \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y\right]_{-1}^2$

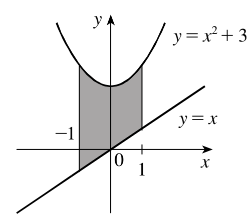
$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = \frac{9}{2}$



17. $A = \int_{-1}^{1} \left[\left(x^2+3\right) - x\right] dx = \int_{-1}^{1} \left(x^2 - x + 3\right) dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x\right]_{-1}^1$

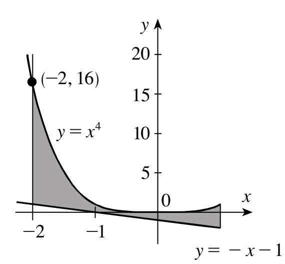
$= \left(\frac{1}{3} - \frac{1}{2} + 3\right) - \left(-\frac{1}{3} - \frac{1}{2} - 3\right) = \frac{20}{3}$



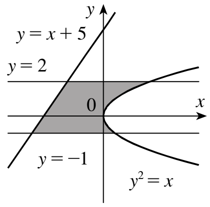
18. $A = \int_{-2}^{0} \left[x^4 - (-x - 1)\right] dx = \int_{-2}^{0} \left(x^4 + x + 1\right) dx$

$= \left[\frac{1}{5}x^5 + \frac{1}{2}x^2 + x\right]_{-2}^0 = 0 - \left(-\frac{32}{5} + 2 - 2\right)$

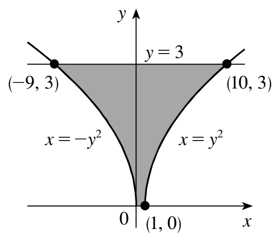
$= \frac{32}{5}$



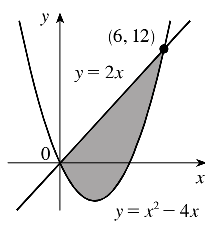
19. $A = \int_{-1}^{2} \left[y^2 - (y-5)\right] dy = \left[\frac{1}{3}y^3 - \frac{1}{2}y^2 + 5y\right]_{-1}^{2}$
$= \left(\frac{8}{3} - 2 + 10\right) - \left(-\frac{1}{3} - \frac{1}{2} - 5\right) = \frac{33}{2}$



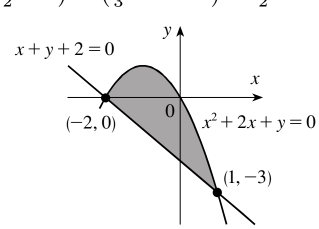
20. $A = \int_0^3 \left[\left(y^2+1\right) - \left(-y^2\right)\right] dy = \int_0^3 \left(2y^2+1\right) dy$
$= \left[\frac{2}{3}y^3 + y\right]_0^3 = 18 + 3 = 21$



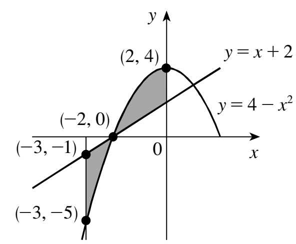
21. $A = \int_0^6 \left[2x - \left(x^2 - 4x\right)\right] dx = \int_0^6 \left(6x - x^2\right) dx$
$= \left[3x^2 - \frac{1}{3}x^3\right]_0^6 = 108 - 72 = 36$



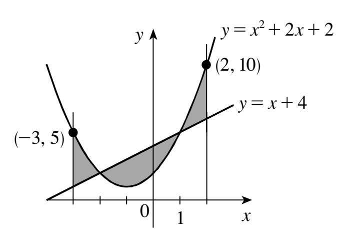
22. $A = \int_{-2}^{1} \left[-x^2 - 2x - (-x-2)\right] dx$
$= \int_{-2}^{1} \left(-x^2 - x + 2\right) dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1}$
$= \left(-\frac{1}{3} - \frac{1}{2} - 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$



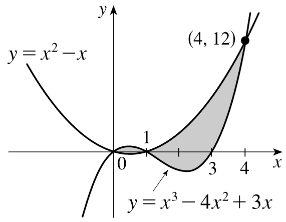
23. $A = \int_{-3}^{0} \left|\left(4 - x^2\right) - (x+2)\right| dx$
$= \int_{-3}^{-2} \left[(x+2) - \left(4 - x^2\right)\right] dx$
$+ \int_{-2}^{0} \left[\left(4 - x^2\right) - (x+2)\right] dx$
$= \int_{-3}^{-2} \left(x^2 + x - 2\right) dx + \int_{-2}^{0} \left(-x^2 - x + 2\right) dx$
$= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_{-3}^{-2}$
$+ \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{0}$
$= \left(-\frac{8}{3} + 2 + 4\right) - \left(-9 + \frac{9}{2} + 6\right)$
$+ 0 - \left(\frac{8}{3} - 2 - 4\right) = \frac{31}{6}$



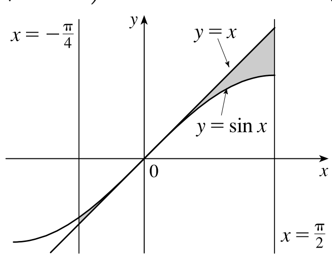
24. $A = \int_{-3}^{-2} \left[\left(x^2 + 2x + 2\right) - (x+4)\right] dx$
$+ \int_{-2}^{1} \left[(x+4) - \left(x^2 + 2x + 2\right)\right] dx$
$+ \int_{1}^{2} \left[\left(x^2 + 2x + 2\right) - (x+4)\right] dx$
$= \int_{-3}^{-2} \left(x^2 + x - 2\right) dx + \int_{-2}^{1} \left(-x^2 - x + 2\right) dx$
$+ \int_{1}^{2} \left(x^2 + x - 2\right) dx$
$= \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_{-3}^{-2} + \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1}$
$+ \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_{1}^{2}$
$= \left[\left(-\frac{8}{3} + 2 + 4\right) - \left(-9 + \frac{9}{2} + 6\right)\right]$
$+ \left[\left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)\right]$
$+ \left[\left(\frac{8}{3} + 2 - 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)\right] = \frac{49}{6}$



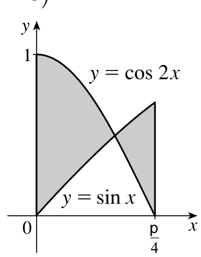
25. $A = \int_0^1 \left[\left(x^3 - 4x^2 + 3x\right) - \left(x^2 - x\right)\right] dx$
$\qquad + \int_1^4 \left[x^2 - x - \left(x^3 - 4x^2 + 3x\right)\right] dx$
$= \int_0^1 \left(x^3 - 5x^2 + 4x\right) dx + \int_1^4 \left(-x^3 + 5x^2 - 4x\right) dx$
$= \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 2x^2\right]_0^1 + \left[-\frac{1}{4}x^4 + \frac{5}{3}x^3 - 2x^2\right]_1^4$
$= \left(\frac{1}{4} - \frac{5}{3} + 2\right) - 0 + \left(-64 + \frac{320}{3} - 32\right)$
$\qquad - \left(-\frac{1}{4} + \frac{5}{3} - 2\right) = \frac{71}{6}$



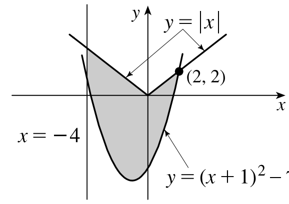
26. $A = \int_{-\pi/4}^{0} (\sin x - x)\, dx + \int_0^{\pi/2} (x - \sin x)\, dx$
$= \left[-\cos x - \frac{1}{2}x^2\right]_{-\pi/4}^{0} + \left[\frac{1}{2}x^2 + \cos x\right]_0^{\pi/2}$
$= 1 - \left(-\frac{1}{\sqrt{2}} - \frac{\pi^2}{32}\right) + \frac{\pi^2}{8} - 1 = \frac{5}{32}\pi^2 + \frac{1}{\sqrt{2}} - 2$



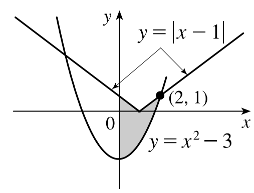
27. $\sin x = \cos 2x = 1 - 2\sin^2 x \quad \Leftrightarrow$
$2\sin^2 x + \sin x - 1 = 0 \quad \Leftrightarrow$
$(2\sin x - 1)(\sin x + 1) = 0 \quad \Leftrightarrow \quad \sin x = \frac{1}{2}$ or $-1 \quad \Leftrightarrow$
$x = \frac{\pi}{6}$.
$A = \int_0^{\pi/6} (\cos 2x - \sin x)\, dx + \int_{\pi/6}^{\pi/4} (\sin x - \cos 2x)\, dx$
$= \left[\frac{1}{2}\sin 2x + \cos x\right]_0^{\pi/6} - \left[\frac{1}{2}\sin 2x + \cos x\right]_{\pi/6}^{\pi/4}$
$= \left(\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) - 1 - 1\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$
$= \frac{1}{2}\left(3\sqrt{3} - \sqrt{2} - 3\right)$



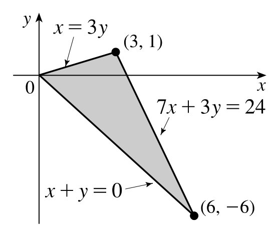
28. $A = \int_{-4}^{0} \left(-x - \left[(x+1)^2 - 7\right]\right) dx$
$\qquad + \int_0^2 \left(x - \left[(x+1)^2 - 7\right]\right) dx$
$= \int_{-4}^{0} \left(-x^2 - 3x + 6\right) dx + \int_0^2 \left(-x^2 - x + 6\right) dx$
$= \left[-\frac{1}{3}x^3 - \frac{3}{2}x^2 + 6x\right]_{-4}^{0} + \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\right]_0^2$
$= 0 - \left(\frac{64}{3} - 24 - 24\right) + \left(-\frac{8}{3} - 2 + 12\right) - 0 = 34$



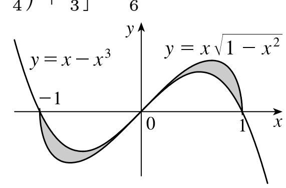
29. $A = \int_0^1 \left[(1 - x) - \left(x^2 - 3\right)\right] dx$
$\qquad + \int_1^2 \left[(x - 1) - \left(x^2 - 3\right)\right] dx$
$= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_0^1 + \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right]_1^2$
$= \left(-\frac{1}{3} - \frac{1}{2} + 4\right) - 0 + \left(-\frac{8}{3} + 2 + 4\right)$
$\qquad - \left(-\frac{1}{3} + \frac{1}{2} + 2\right) = \frac{13}{3}$



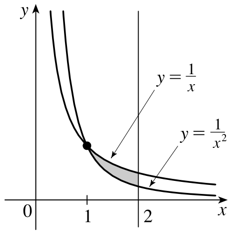
30. $A = \int_0^3 \left[\frac{1}{3}x - (-x)\right] dx + \int_3^6 \left[\left(8 - \frac{7}{3}x\right) - (-x)\right] dx$
$= \int_0^3 \frac{4}{3}x\, dx + \int_3^6 \left(-\frac{4}{3}x + 8\right) dx$
$= \left[\frac{2}{3}x^2\right]_0^3 + \left[-\frac{2}{3}x^2 + 8x\right]_3^6$
$= (6 - 0) + (24 - 18) = 12$



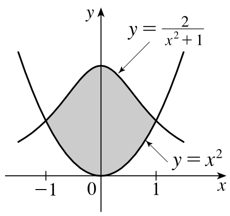
31. $A = \int_{-1}^{1} \left|x\sqrt{1 - x^2} - \left(x - x^3\right)\right| dx$
$= \int_{-1}^{0} \left(x - x^3 - x\sqrt{1 - x^2}\right) dx$
$\qquad + \int_0^1 \left(x\sqrt{1 - x^2} - x + x^3\right) dx$
$= 2\int_0^1 \left(x\sqrt{1 - x^2} - x + x^3\right) dx$ (by symmetry)
$= 2\left[-\frac{1}{3}\left(1 - x^2\right)^{3/2} - \frac{1}{2}x^2 + \frac{1}{4}x^4\right]_0^1$
$= 2\left[\left(-\frac{1}{2} + \frac{1}{4}\right) + \frac{1}{3}\right] = \frac{1}{6}$



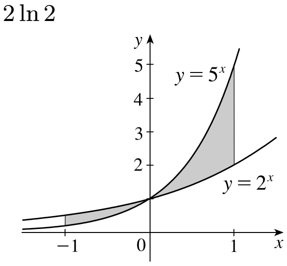
32. $A = \int_1^2 \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \left[\ln x + \frac{1}{x}\right]_1^2$
$= \left(\ln 2 + \frac{1}{2}\right) - (\ln 1 + 1) = \ln 2 - \frac{1}{2}$



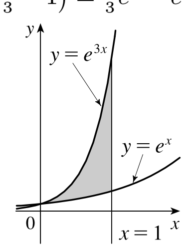
33. $A = 2\int_0^1 \left(\frac{2}{x^2+1} - x^2\right) dx = \left[4\tan^{-1} x - \frac{2}{3}x^3\right]_0^1$
$= 4 \cdot \frac{\pi}{4} - \frac{2}{3} = \pi - \frac{2}{3}$



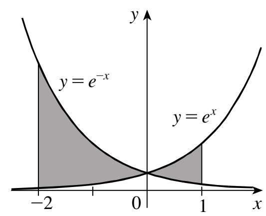
34. $A = \int_{-1}^0 (2^x - 5^x)\, dx + \int_0^1 (5^x - 2^x)\, dx$
$= \left[\frac{2^x}{\ln 2} - \frac{5^x}{\ln 5}\right]_{-1}^0 + \left[\frac{5^x}{\ln 5} - \frac{2^x}{\ln 2}\right]_0^1$
$= \left(\frac{1}{\ln 2} - \frac{1}{\ln 5}\right) - \left(\frac{1/2}{\ln 2} - \frac{1/5}{\ln 5}\right)$
$+ \left(\frac{5}{\ln 5} - \frac{2}{\ln 2}\right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 2}\right)$
$= \frac{16}{5\ln 5} - \frac{1}{2\ln 2}$



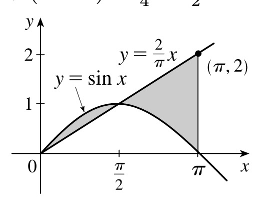
35. $A = \int_0^1 \left(e^{3x} - e^x\right) dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1$
$= \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3}$



36. $A = \int_{-2}^0 \left(e^{-x} - e^x\right) dx + \int_0^1 \left(e^x - e^{-x}\right) dx$
$= \left[-e^{-x} - e^x\right]_{-2}^0 + \left[e^x + e^{-x}\right]_0^1$
$= (-1 - 1) - \left(-e^2 - e^{-2}\right)$
$+ \left(e + e^{-1}\right) - (1 + 1)$
$= e^2 + e + e^{-1} + e^{-2} - 4$



37. $\int_0^\pi \left|\sin x - \frac{2}{\pi}x\right| dx$
$= \int_0^{\pi/2} \left(\sin x - \frac{2}{\pi}x\right) dx + \int_{\pi/2}^{\pi} \left(\frac{2}{\pi}x - \sin x\right) dx$
$= \left[-\cos x - \frac{x^2}{\pi}\right]_0^{\pi/2} + \left[\frac{x^2}{\pi} + \cos x\right]_{\pi/2}^{\pi}$
$= -\frac{\pi}{4} + 1 + (\pi - 1) - \frac{\pi}{4} = \frac{\pi}{2}$



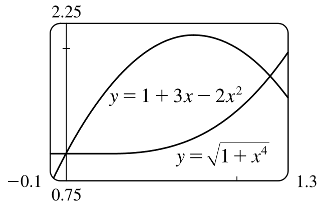
38. Let $f(x) = \sqrt{1+x^3} - (1-x)$, $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.

$$A = \int_0^2 \left[\sqrt{1+x^3} - (1-x)\right] dx$$
$$\approx \tfrac{1}{2}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right]$$
$$= \tfrac{1}{2}\left[\left(\tfrac{\sqrt{65}}{8} - \tfrac{3}{4}\right) + \left(\tfrac{\sqrt{91}}{8} - \tfrac{1}{4}\right) + \left(\tfrac{3\sqrt{21}}{8} + \tfrac{1}{4}\right) + \left(\tfrac{\sqrt{407}}{8} + \tfrac{3}{4}\right)\right]$$
$$= \tfrac{1}{16}\left(\sqrt{65} + \sqrt{91} + 3\sqrt{21} + \sqrt{407}\right) \approx 3.22$$

39. Let $f(x) = x - x\tan x$, and $\Delta x = \frac{\pi/4 - 0}{4} = \frac{\pi}{16}$. Then

$$A = \int_0^{\pi/4} (x - x\tan x)\, dx$$
$$\approx \tfrac{\pi}{16}\left[f\left(\tfrac{\pi}{32}\right) + f\left(\tfrac{3\pi}{32}\right) + f\left(\tfrac{5\pi}{32}\right) + f\left(\tfrac{7\pi}{32}\right)\right]$$
$$\approx \tfrac{\pi}{16}\left[\tfrac{\pi}{32}\left(1 - \tan\tfrac{\pi}{32}\right) + \tfrac{3\pi}{32}\left(1 - \tan\tfrac{3\pi}{32}\right) + \tfrac{5\pi}{32}\left(1 - \tan\tfrac{5\pi}{32}\right) + \tfrac{7\pi}{32}\left(1 - \tan\tfrac{7\pi}{32}\right)\right]$$
$$= \tfrac{\pi^2}{512}\left[16 - \tan\tfrac{\pi}{32} - 3\tan\tfrac{3\pi}{32} - 5\tan\tfrac{5\pi}{32} - 7\tan\tfrac{7\pi}{32}\right]$$
$$\approx 0.1267$$

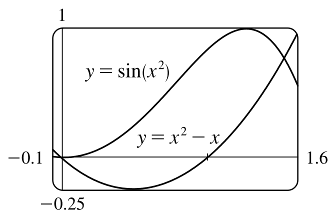
40.



From the graph, we see that the curves intersect at $x = 0$ and at $x \approx 1.19$, with $1 + 3x - 2x^2 > \sqrt{1 + x^4}$ on $(0, 1.19)$. So, using the Midpoint Rule with $f(x) = 1 + 3x - 2x^2 - \sqrt{1 + x^4}$ on $[0, 1.19]$ with $n = 4$, we calculate the approximate area between the curves:

$$\begin{aligned} A &\approx \int_0^{1.19} \left(1 + 3x - 2x^2 - \sqrt{1 + x^4}\right) dx \\ &\approx \tfrac{1.19}{4}\left[f\left(\tfrac{1.19}{8}\right) + f\left(\tfrac{3 \cdot 1.19}{8}\right) \right. \\ &\qquad \left. + f\left(\tfrac{5 \cdot 1.19}{8}\right) + f\left(\tfrac{7 \cdot 1.19}{8}\right)\right] \approx 0.83 \end{aligned}$$

41.



From the graph, we see that the curves intersect at $x = 0$ and at $x \approx 1.5$, with $\sin\left(x^2\right) > x^2 - x$ on $(0, 1.51)$. So, using the Midpoint Rule with $f(x) = \sin\left(x^2\right) - x^2 + x$ on $(0, 1.51)$ with $n = 4$, we calculate that the area between the curves is

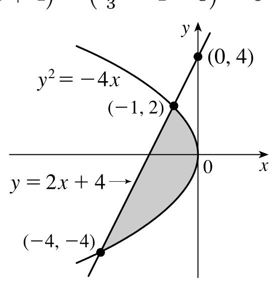
$$\begin{aligned} A &\approx \int_0^{1.51} \left[\sin\left(x^2\right) - \left(x^2 - x\right)\right] dx \\ &\approx \tfrac{1.51}{4}\left[f\left(\tfrac{1.51}{8}\right) + f\left(\tfrac{3 \cdot 1.51}{8}\right) \right. \\ &\qquad \left. + f\left(\tfrac{5 \cdot 1.51}{8}\right) + f\left(\tfrac{7 \cdot 1.51}{8}\right)\right] \approx 0.81 \end{aligned}$$

42. (a) $A = \int_{-4}^{-1} \left[(2x + 4) + \sqrt{-4x}\right] dx + \int_{-1}^{0} 2\sqrt{-4x}\, dx$

$$\begin{aligned} &= \left[x^2 + 4x\right]_{-4}^{-1} + 2\int_{-4}^{-1} \sqrt{-x}\, dx + 4\int_{-1}^{0} \sqrt{-x}\, dx \\ &= (-3 - 0) + 2\int_1^4 \sqrt{u}\, du + 4\int_0^1 \sqrt{u}\, du \quad (u = -x) \\ &= -3 + \left[\tfrac{4}{3}u^{3/2}\right]_1^4 + \left[\tfrac{8}{3}u^{3/2}\right]_0^1 \\ &= -3 + \tfrac{28}{3} + \tfrac{8}{3} = 9 \end{aligned}$$

(b) $A = \int_{-4}^{2} \left[-\frac{1}{4}y^2 - \left(\frac{1}{2}y - 2\right)\right] dy$

$$\begin{aligned} &= \left[-\tfrac{1}{12}y^3 - \tfrac{1}{4}y^2 + 2y\right]_{-4}^{2} \\ &= \left(-\tfrac{2}{3} - 1 + 4\right) - \left(\tfrac{16}{3} - 4 - 8\right) = 9 \end{aligned}$$

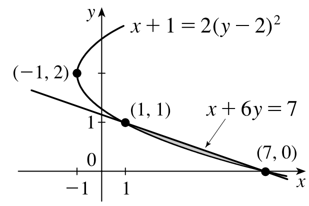


43. (a) $A = \int_1^7 \left[\left(\frac{7}{6} - \frac{1}{6}x\right) - \left(2 - \sqrt{\frac{1}{2}(x + 1)}\right)\right] dx$

$$\begin{aligned} &= \int_1^7 \left[-\tfrac{5}{6} - \tfrac{1}{6}x + \tfrac{1}{\sqrt{2}}(x + 1)^{1/2}\right] dx \\ &= \left[-\tfrac{5}{6}x - \tfrac{1}{12}x^2 + \tfrac{1}{\sqrt{2}} \cdot \tfrac{2}{3}(x + 1)^{3/2}\right]_1^7 \\ &= -5 - 4 + \tfrac{\sqrt{2}}{3}\left(8\sqrt{8} - 2\sqrt{2}\right) = \tfrac{1}{3} \end{aligned}$$

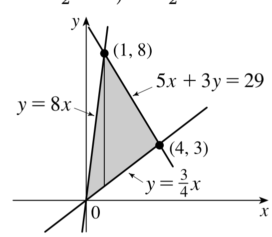
(b) $A = \int_0^1 \left((7 - 6y) - \left[2(y - 2)^2 - 1\right]\right) dy$

$$\begin{aligned} &= \int_0^1 \left(-2y^2 + 2y\right) dy = \left[-\tfrac{2}{3}y^3 + y^2\right]_0^1 \\ &= -\tfrac{2}{3} + 1 = \tfrac{1}{3} \end{aligned}$$



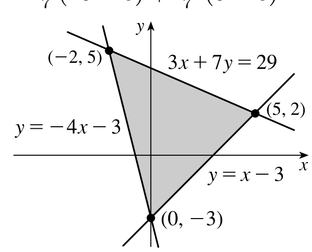
44. $A = \int_0^1 \left(8x - \frac{3}{4}x\right) dx + \int_1^4 \left[\left(-\frac{5}{3}x + \frac{29}{3}\right) - \frac{3}{4}x\right] dx$

$$\begin{aligned} &= \tfrac{29}{4}\int_0^1 x\, dx + \int_1^4 \left(-\tfrac{29}{12}x + \tfrac{29}{3}\right) dx \\ &= \tfrac{29}{4}\left[\tfrac{1}{2}x^2\right]_0^1 - \tfrac{29}{12}\left[\tfrac{1}{2}x^2 - 4x\right]_1^4 \\ &= \tfrac{29}{8} - \tfrac{29}{12}\left(-8 - \tfrac{1}{2} + 4\right) = \tfrac{29}{2} \end{aligned}$$

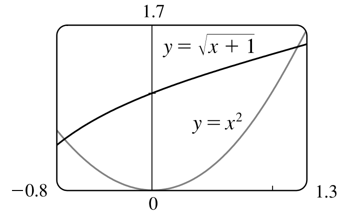


45. $A = \int_{-2}^{0} \left[\left(-\frac{3}{7}x + \frac{29}{7}\right) - (-4x - 3)\right] dx$

$$\begin{aligned} &\qquad + \int_0^5 \left[\left(-\tfrac{3}{7}x + \tfrac{29}{7}\right) - (x - 3)\right] dx \\ &= \int_{-2}^{0} \left[\tfrac{25}{7}x + \tfrac{50}{7}\right] dx + \int_0^5 \left[-\tfrac{10}{7}x + \tfrac{50}{7}\right] dx \\ &= \left[\tfrac{25}{7}\left(\tfrac{1}{2}x^2\right) + \tfrac{50}{7}x\right]_{-2}^{0} + \left[-\tfrac{5}{7}x^2 + \tfrac{50}{7}x\right]_0^5 \\ &= \tfrac{25}{7}(0 - 2) + \tfrac{50}{7}(0 + 2) \\ &\qquad - \tfrac{5}{7}(25 - 0) + \tfrac{50}{7}(5 - 0) = 25 \end{aligned}$$



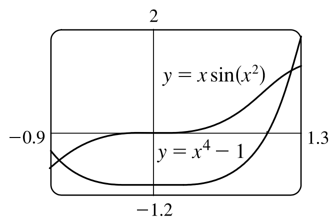
46.



From the graph, we see that the curves intersect at $x \approx -0.72$ and at $x \approx 1.22$, with $\sqrt{x+1} > x^2$ on $[-0.72, 1.22]$. So the area between the curves is

$$\begin{aligned} A &\approx \textstyle\int_{-0.72}^{1.22} \left(\sqrt{x+1} - x^2\right) dx \\ &= \left[\tfrac{2}{3}(x+1)^{3/2} - \tfrac{1}{3}x^3\right]_{-0.72}^{1.22} \\ &\approx 1.38 \end{aligned}$$

47.



From the graph, we see that the curves intersect at $x \approx -0.83$ and $x \approx 1.22$, with $x\sin\left(x^2\right) > x^4 - 1$ on $[-0.83, 1.22]$. So the area between the curves is

$$\begin{aligned} A &\approx \textstyle\int_{-0.83}^{1.22} \left[x\sin\left(x^2\right) - \left(x^4 - 1\right)\right] dx \\ &= \left[-\tfrac{1}{2}\cos\left(x^2\right) - \tfrac{1}{5}x^5 + x\right]_{-0.83}^{1.22} \\ &\approx 1.78 \end{aligned}$$

48. A typical graphing calculator solution is as follows. Assign `X^2` to `Y1` ($y_1 = x^2$) and `Exp(-X^2)` to `Y2` ($y_2 = e^{-x^2}$). Graph the functions and find (and store) the x-coordinates of the points of intersection. In this case, we have some symmetry, so we need to find only one point of intersection. Store $x \approx 0.75308916$ in memory location `B`. Now use the appropriate integration command to approximate the area:

$$\begin{aligned} A &= 2\textstyle\int_0^B (y_2 - y_1)\, dx \\ &= \texttt{2*Int(Y2-Y1,X,0,B)} \\ &\approx 0.979263 \end{aligned}$$