

7.2 VOLUMES

A Click here for answers.

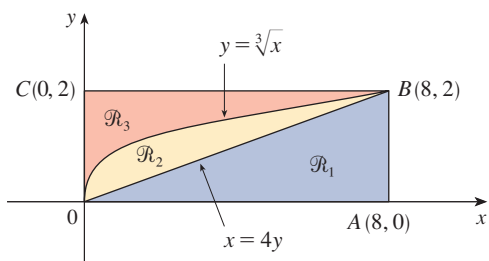
1–5 ■ Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

- $y^2 = x^3$, $x = 4$, $y = 0$; about the x -axis
- $x + y = 1$, $x = 0$, $y = 0$; about the x -axis
- $y = x^2$, $y = 4$, $x = 0$, $x = 2$; about the y -axis
- $y = x^2 + 1$, $y = 3 - x^2$; about the x -axis
- $y = 2x - x^2$, $y = 0$, $x = 0$, $x = 1$; about the y -axis

6–13 ■ Find the volume of the solid obtained by rotating the region bounded by the given curves about the x -axis.

- $y = x^2 - 1$, $y = 0$, $x = 0$, $x = 2$
- $y = -1/x$, $y = 0$, $x = 1$, $x = 3$
- $y = e^x$, $y = 0$, $x = 0$, $x = 1$
- $y = 1/\sqrt{x+1}$, $y = 0$, $x = 0$, $x = 1$
- $y = \sec x$, $y = 1$, $x = -1$, $x = 1$
- $y = \cos x$, $y = \sin x$, $x = 0$, $x = \pi/4$
- $y = |x + 2|$, $y = 0$, $x = -3$, $x = 0$
- $y = \lfloor x \rfloor$, $x = 1$, $x = 6$, $y = 0$

14–25 ■ Refer to the figure and find the volume generated by rotating the given region about the given line.



- | | |
|----------------------|----------------------|
| 14. R_1 about OA | 15. R_1 about OC |
| 16. R_1 about AB | 17. R_1 about BC |
| 18. R_2 about OA | 19. R_2 about OC |
| 20. R_2 about BC | 21. R_2 about AB |
| 22. R_3 about OA | 23. R_3 about OC |
| 24. R_3 about BC | 25. R_3 about AB |

S Click here for solutions.

26–30 ■ Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

- $y = \ln x$, $y = 1$, $x = 1$; about the x -axis
- $y = \sqrt{x-1}$, $y = 0$, $x = 5$; about the y -axis
- $x - y = 1$, $y = (x-4)^2 + 1$; about $y = 7$
- $y = \cos x$, $y = 0$, $x = 0$, $x = \pi/2$; about $y = 1$
- $y = \cos x$, $y = 0$, $x = 0$, $x = \pi/2$; about $y = -1$

31–32 ■ Use a graph to find approximate x -coordinates of the points of intersection of the given curves. Then find (approximately) the volume of the solid obtained by rotating about the x -axis the region bounded by these curves.

- $y = x^2$, $y = \sqrt{x+1}$
- $y = x^4$, $y = 3x - x^3$

33–34 ■ Sketch and find the volume of the solid obtained by rotating the region under the graph of f about the x -axis.

$$33. f(x) = \begin{cases} 3 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x < 4 \\ 3 & \text{if } 4 \leq x \leq 5 \end{cases}$$

$$34. f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x < 1 \\ x^2 - 2x + 2 & \text{if } 1 \leq x \leq 2 \end{cases}$$

35–40 ■ Each integral represents the volume of a solid. Describe the solid.

$$35. \pi \int_0^{\pi/4} \tan^2 x \, dx$$

$$36. \pi \int_1^2 y^6 \, dy$$

$$37. \pi \int_0^1 (y - y^2) \, dy$$

$$38. \pi \int_0^4 [16 - (x-2)^4] \, dx$$

$$39. \pi \int_0^1 [(5-2x^2)^2 - (5-2x)^2] \, dx$$

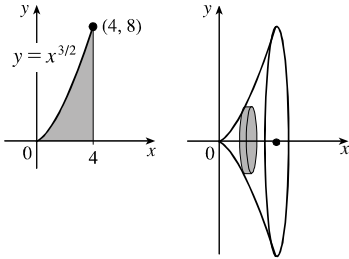
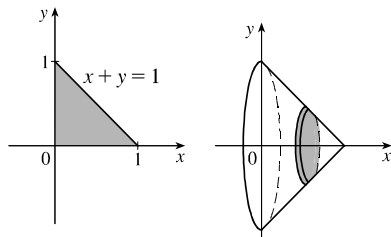
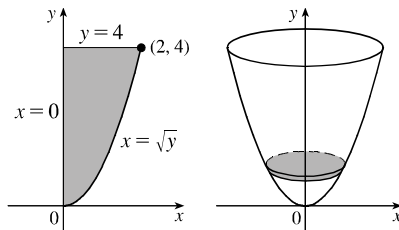
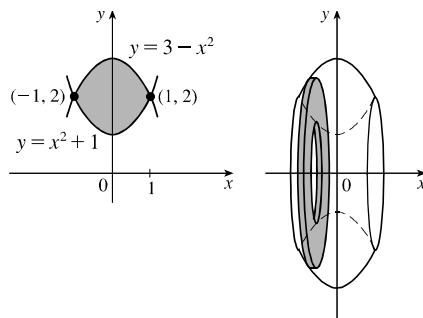
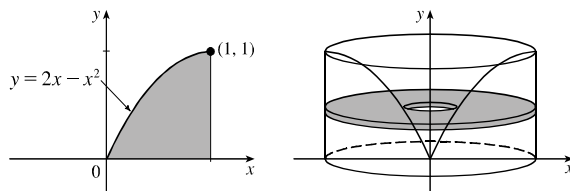
$$40. \pi \int_{\pi/4}^{\pi/2} [(2 + \sin x)^2 - (2 + \cos x)^2] \, dx$$

- 41.** The base of S is the triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$. Cross-sections perpendicular to the x -axis are semi-circles. Find the volume of S .

7.2 ANSWERS

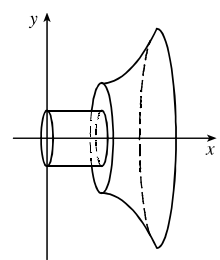
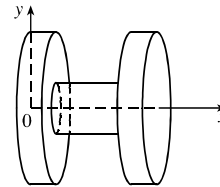
E Click here for exercises.

S Click here for solutions.

1. 64π 2. $\frac{\pi}{3}$ 3. 8π 4. $\frac{32}{3}\pi$ 5. $\frac{5}{6}\pi$ 6. $\frac{46}{15}\pi$ 8. $\frac{\pi}{2}(e^2 - 1)$ 10. $2\pi(\tan 1 - 1)$ 12. 3π 14. $\frac{32}{3}\pi$ 7. $\frac{2}{3}\pi$ 9. $\pi \ln 2$ 11. $\frac{\pi}{2}$ 13. 55π 15. $\frac{256}{3}\pi$ 16. $\frac{128}{3}\pi$ 18. $\frac{128}{15}\pi$ 20. $\frac{112}{15}\pi$ 22. $\frac{64}{5}\pi$ 24. $\frac{16}{5}\pi$ 17. $\frac{64}{3}\pi$ 19. $\frac{512}{21}\pi$ 21. $\frac{832}{21}\pi$ 23. $\frac{128}{7}\pi$ 25. $\frac{320}{7}\pi$ 26. $V = \pi \int_1^e [1^2 - (\ln x)^2] dx$ 27. $V = \pi \int_0^2 (24 - y^4 - 2y^2) dy$ 28. $V = \pi \int_3^6 (x^4 - 16x^3 + 83x^2 - 144x + 36) dx$ 29. $V = \pi \int_0^{\pi/2} (2 \cos x - \cos^2 x) dx$ 30. $V = \pi \int_0^{\pi/2} (2 \cos x + \cos^2 x) dx$

31. 5.80

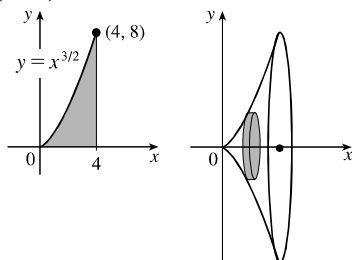
32. 6.74

33. 21π 34. $\frac{127\pi}{60}$ 35. Solid obtained by rotating the region under the curve $y = \tan x$, from $x = 0$ to $x = \frac{\pi}{4}$, about the x -axis36. Solid obtained by rotating the region bounded by the curve $x = y^3$ and the lines $y = 1$, $y = 2$, and $x = 0$ about the y -axis37. Solid obtained by rotating the region between the curves $x = y$ and $x = \sqrt{y}$ about the y -axis38. Solid obtained by rotating the region bounded by the curve $y = (x - 2)^2$ and the line $y = 4$ about the x -axis39. Solid obtained by rotating the region between the curves $y = 5 - 2x^2$ and $y = 5 - 2x$ about the x -axis. Or: Solid obtained by rotating the region bounded by the curves $y = 2x$ and $y = 2x^2$ about the line $y = 5$ 40. Solid obtained by rotating the region bounded by the curves $y = 2 + \cos x$ and $y = 2 + \sin x$ and the line $x = \frac{\pi}{2}$ about the x -axis41. $\frac{\pi}{12}$

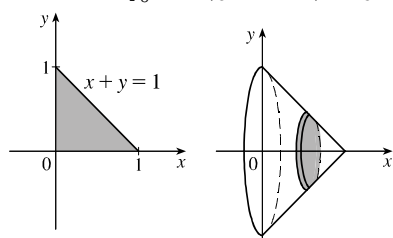
7.2 SOLUTIONS

Click here for exercises.

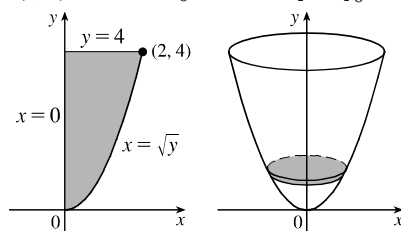
$$1. V = \int_0^4 \pi \left(x^{3/2}\right)^2 dx = \pi \left[\frac{1}{4}x^4\right]_0^4 = 64\pi$$



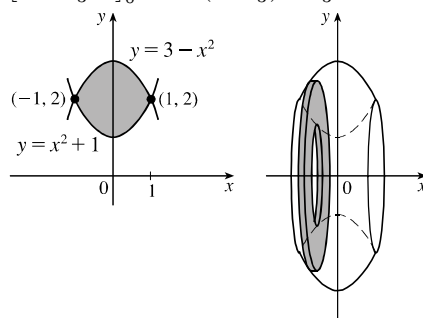
$$2. V = \int_0^1 \pi (-x+1)^2 dx = \pi \int_0^1 (x^2 - 2x + 1) dx \\ = \pi \left[\frac{1}{3}x^3 - x^2 + x\right]_0^1 = \pi \left(\frac{1}{3} - 1 + 1\right) = \frac{\pi}{3}$$



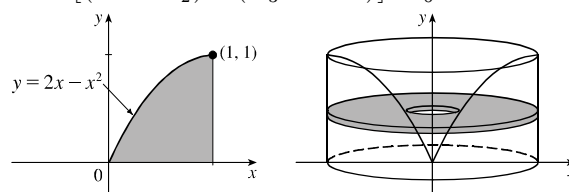
$$3. V = \int_0^4 \pi (\sqrt{y})^2 dy = \pi \int_0^4 y dy = \pi \left[\frac{1}{2}y^2\right]_0^4 = 8\pi$$



$$4. V = \pi \int_{-1}^1 \left[(3-x^2)^2 - (x^2+1)^2\right] dx \\ = \pi \int_{-1}^1 (8-8x^2) dx = 2\pi \int_0^1 (8-8x^2) dx \\ = 2\pi \left[8x - \frac{8}{3}x^3\right]_0^1 = 2\pi \left(8 - \frac{8}{3}\right) = \frac{32}{3}\pi$$



$$5. V = \pi \int_0^1 \left[1^2 - (1 - \sqrt{1-y})^2\right] dy \\ = \pi \int_0^1 (2\sqrt{1-y} - 1 + y) dy \\ = \pi \left[-\frac{4}{3}(1-y)^{3/2} - y + \frac{1}{2}y^2\right]_0^1 \\ = \pi \left[(0 - 1 + \frac{1}{2}) - (-\frac{4}{3} - 0 + 0)\right] = \frac{5}{6}\pi$$



$$6. V = \pi \int_0^2 (x^2 - 1)^2 dx = \pi \int_0^2 (x^4 - 2x^2 + 1) dx \\ = \pi \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_0^2 = \pi \left(\frac{32}{5} - \frac{16}{3} + 2\right) = \frac{46}{15}\pi$$

$$7. V = \pi \int_1^3 (1/x)^2 dx = \pi [-1/x]_1^3 = \pi \left(-\frac{1}{3} + 1\right) = \frac{2}{3}\pi$$

$$8. V = \int_0^1 \pi (e^x)^2 dx = \int_0^1 \pi e^{2x} dx = \frac{1}{2} [\pi e^{2x}]_0^1 \\ = \frac{\pi}{2} (e^2 - 1)$$

9. The cross-sectional area is

$$\pi \left(1/\sqrt{x+1}\right)^2 = \pi/(x+1). \text{ Therefore, the volume is } \\ \int_0^1 \frac{\pi}{x+1} dx = \pi [\ln(x+1)]_0^1 = \pi \ln 2 - \ln 1 = \pi \ln 2.$$

$$10. V = \pi \int_{-1}^1 (\sec^2 x - 1^2) dx = \pi [\tan x - x]_{-1}^1 \\ = \pi [(\tan 1 - 1) - (-\tan 1 + 1)] = 2\pi (\tan 1 - 1)$$

$$11. V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) dx \\ = \frac{\pi}{2} \int_0^{\pi/4} \cos 2x (2 dx) = \frac{\pi}{2} [\sin 2x]_0^{\pi/4} \\ = \frac{\pi}{2} (1 - 0) = \frac{\pi}{2}$$

$$12. V = \pi \int_{-3}^{-2} (-x-2)^2 dx + \pi \int_{-2}^0 (x+2)^2 dx \\ = \pi \int_{-3}^0 (x+2)^2 dx = \left[\frac{\pi}{3}(x+2)^3\right]_{-3}^0 \\ = \frac{\pi}{3} [8 - (-1)] = 3\pi$$

$$13. V = \pi \int_1^2 1^2 dx + \pi \int_2^3 2^2 dx + \pi \int_3^4 3^2 dx \\ + \pi \int_4^5 4^2 dx + \pi \int_5^6 5^2 dx \\ = \pi \cdot 1 + \pi \cdot 4 + \pi \cdot 9 + \pi \cdot 16 + \pi \cdot 25 = 55\pi$$

$$14. V = \pi \int_0^8 \left(\frac{1}{4}x\right)^2 dx = \frac{\pi}{16} \left[\frac{1}{3}x^3\right]_0^8 = \frac{32}{3}\pi$$

$$15. V = \pi \int_0^2 [8^2 - (4y)^2] dy = \pi [64y - \frac{16}{3}y^3]_0^2 \\ = \pi \left(128 - \frac{128}{3}\right) = \frac{256}{3}\pi$$

$$16. V = \pi \int_0^2 (8-4y)^2 dy = \pi [64y - 32y^2 + \frac{16}{3}y^3]_0^2 \\ = \pi \left(128 - 128 + \frac{128}{3}\right) = \frac{128}{3}\pi$$

$$\begin{aligned}
 17. V &= \pi \int_0^8 \left[2^2 - \left(2 - \frac{1}{4}x \right)^2 \right] dx = \pi \int_0^8 \left(x - \frac{1}{16}x^2 \right) dx \\
 &= \pi \left[\frac{1}{2}x^2 - \frac{1}{48}x^3 \right]_0^8 = \pi \left(32 - \frac{32}{3} \right) = \frac{64}{3}\pi
 \end{aligned}$$

$$\begin{aligned}
 18. V &= \pi \int_0^8 \left[(\sqrt[3]{x})^2 - \left(\frac{1}{4}x \right)^2 \right] dx = \pi \int_0^8 \left(x^{2/3} - \frac{1}{16}x^2 \right) dx \\
 &= \pi \left[\frac{3}{5}x^{5/3} - \frac{1}{48}x^3 \right]_0^8 = \pi \left(\frac{96}{5} - \frac{32}{3} \right) = \frac{128}{15}\pi
 \end{aligned}$$

$$\begin{aligned}
 19. V &= \pi \int_0^2 \left[(4y)^2 - (y^3)^2 \right] dy = \pi \int_0^2 (16y^2 - y^6) dy \\
 &= \pi \left[\frac{16}{3}y^3 - \frac{1}{7}y^7 \right]_0^2 = \pi \left(\frac{128}{3} - \frac{128}{7} \right) = \frac{512}{21}\pi
 \end{aligned}$$

$$\begin{aligned}
 20. V &= \pi \int_0^8 \left[\left(2 - \frac{1}{4}x \right)^2 - (2 - \sqrt[3]{x})^2 \right] dx \\
 &= \pi \int_0^8 \left(-x + \frac{1}{16}x^2 + 4x^{1/3} - x^{2/3} \right) dx \\
 &= \pi \left[-\frac{1}{2}x^2 + \frac{1}{48}x^3 + 3x^{4/3} - \frac{3}{5}x^{5/3} \right]_0^8 \\
 &= \pi \left(-32 + \frac{32}{3} + 48 - \frac{96}{5} \right) = \frac{112}{15}\pi
 \end{aligned}$$

$$\begin{aligned}
 21. V &= \pi \int_0^2 \left[(8 - y^3)^2 - (8 - 4y)^2 \right] dy \\
 &= \pi \int_0^2 (-16y^3 + y^6 + 64y - 16y^2) dy \\
 &= \pi \left[-4y^4 + \frac{1}{7}y^7 + 32y^2 - \frac{16}{3}y^3 \right]_0^2 \\
 &= \pi \left(-64 + \frac{128}{7} + 128 - \frac{128}{3} \right) = \frac{832}{21}\pi
 \end{aligned}$$

$$\begin{aligned}
 22. V &= \pi \int_0^8 (2^2 - x^{2/3}) dx = \pi \left[4x - \frac{3}{5}x^{5/3} \right]_0^8 \\
 &= \pi \left(32 - \frac{96}{5} \right) = \frac{64}{5}\pi
 \end{aligned}$$

$$23. V = \pi \int_0^2 (y^3)^2 dy = \pi \left[\frac{1}{7}y^7 \right]_0^2 = \frac{128}{7}\pi$$

$$\begin{aligned}
 24. V &= \pi \int_0^8 (2 - \sqrt[3]{x})^2 dx = \pi \int_0^8 (4 - 4x^{1/3} + x^{2/3}) dx \\
 &= \pi \left[4x - 3x^{4/3} + \frac{3}{5}x^{5/3} \right]_0^8 \\
 &= \pi \left(32 - 48 + \frac{96}{5} \right) = \frac{16}{5}\pi
 \end{aligned}$$

$$\begin{aligned}
 25. V &= \pi \int_0^2 [8^2 - (8 - y^3)^2] dy = \pi \int_0^2 (16y^3 - y^6) dy \\
 &= \pi \left[4y^4 - \frac{1}{7}y^7 \right]_0^2 = \pi \left(64 - \frac{128}{7} \right) = \frac{320}{7}\pi
 \end{aligned}$$

$$26. V = \pi \int_1^e [1^2 - (\ln x)^2] dx$$

$$27. V = \pi \int_0^2 [5^2 - (y^2 + 1)^2] dy = \pi \int_0^2 (24 - y^4 - 2y^2) dy$$

$$\begin{aligned}
 28. x - 1 &= (x - 4)^2 + 1 \Leftrightarrow x^2 - 9x + 18 = 0 \Leftrightarrow \\
 &x = 3 \text{ or } 6, \text{ so} \\
 V &= \pi \int_3^6 [6 - (x - 4)^2]^2 - (8 - x)^2 dx \\
 &= \pi \int_3^6 (x^4 - 16x^3 + 83x^2 - 144x + 36) dx
 \end{aligned}$$

$$\begin{aligned}
 29. V &= \pi \int_0^{\pi/2} [1^2 - (1 - \cos x)^2] dx \\
 &= \pi \int_0^{\pi/2} (2 \cos x - \cos^2 x) dx
 \end{aligned}$$

$$\begin{aligned}
 30. V &= \pi \int_0^{\pi/2} [(1 + \cos x)^2 - 1^2] dx \\
 &= \pi \int_0^{\pi/2} (2 \cos x + \cos^2 x) dx
 \end{aligned}$$

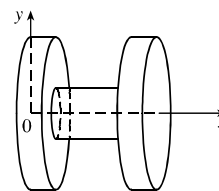
31. We see from the graph in [Archived Problem 7.1.46](#) that the x -coordinates of the points of intersection are $x \approx -0.72$ and $x \approx 1.22$, with $\sqrt{x+1} > x^2$ on $[-0.72, 1.22]$, so the volume of revolution is about

$$\begin{aligned}
 &\pi \int_{-0.72}^{1.22} [(\sqrt{x+1})^2 - (x^2)^2] dx \\
 &= \pi \int_{-0.72}^{1.22} (x+1-x^4) dx \\
 &= \pi \left[\frac{1}{2}x^2 + x - \frac{1}{5}x^5 \right]_{-0.72}^{1.22} \\
 &\approx 5.80
 \end{aligned}$$

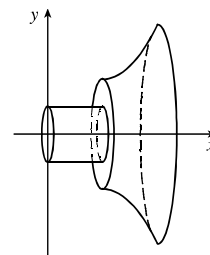
32. The x -coordinates of the points of intersection are $x = 0$ and $x \approx 1.17$, with $3x - x^3 > x^4$ on $[0, 1.17]$, so the volume of revolution is about

$$\begin{aligned}
 &\pi \int_0^{1.17} [(3x - x^3)^2 - (x^4)^2] dx \\
 &= \pi \int_0^{1.17} [9x^2 - 6x^4 + x^6 - x^8] dx \\
 &= \pi \left[3x^3 - \frac{6}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{9}x^9 \right]_0^{1.17} \\
 &\approx 6.74
 \end{aligned}$$

$$\begin{aligned}
 33. V &= \pi \int_0^1 3^2 dx + \pi \int_1^4 1^2 dx + \pi \int_4^5 3^2 dx \\
 &= 9\pi + 3\pi + 9\pi = 21\pi
 \end{aligned}$$



$$\begin{aligned}
 34. V &= \pi \int_0^1 \left(\frac{1}{2} \right)^2 dx + \pi \int_1^2 (x^2 - 2x + 2)^2 dx \\
 &= \frac{\pi}{4} + \pi \int_1^2 (x^4 - 4x^3 + 8x^2 - 8x + 4) dx \\
 &= \frac{\pi}{4} + \pi \left[\frac{1}{5}x^5 - x^4 + \frac{8}{3}x^3 - 4x^2 + 4x \right]_1^2 \\
 &= \frac{\pi}{4} + \pi \left[\left(\frac{32}{5} - 16 + \frac{64}{3} - 16 + 8 \right) - \left(\frac{1}{5} - 1 + \frac{8}{3} - 4 + 4 \right) \right] = \frac{127\pi}{60}
 \end{aligned}$$



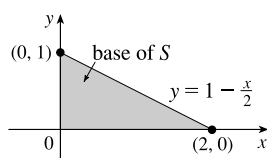
35. The solid is obtained by rotating the region under the curve $y = \tan x$, from $x = 0$ to $x = \frac{\pi}{4}$, about the x -axis.

36. The solid is obtained by rotating the region bounded by the curve $x = y^3$ and the lines $y = 1$, $y = 2$, and $x = 0$ about the y -axis.

37. The solid is obtained by rotating the region between the curves $x = y$ and $x = \sqrt{y}$ about the y -axis.

38. The solid is obtained by rotating the region bounded by the curve $y = (x - 2)^2$ and the line $y = 4$ about the x -axis.
39. The solid is obtained by rotating the region between the curves $y = 5 - 2x^2$ and $y = 5 - 2x$ about the x -axis. Or: The solid is obtained by rotating the region bounded by the curves $y = 2x$ and $y = 2x^2$ about the line $y = 5$.
40. The solid is obtained by rotating the region bounded by the curves $y = 2 + \cos x$ and $y = 2 + \sin x$ and the line $x = \frac{\pi}{2}$ about the x -axis.

41.



Since the area of a semicircle of diameter y is $\frac{\pi y^2}{8}$, we have

$$\begin{aligned}
 V &= \int_0^2 A(x) \, dx = \int_0^2 \frac{\pi}{8} y^2 \, dx \\
 &= \frac{\pi}{8} \int_0^2 \left(1 - \frac{1}{2}x\right)^2 \, dx = \frac{\pi}{4} \int_0^2 \left(\frac{1}{2}x - 1\right)^2 \frac{1}{2} \, dx \\
 &= \frac{\pi}{4} \left[\frac{1}{3} \left(\frac{1}{2}x - 1\right)^3 \right]_0^2 = \frac{\pi}{12} [0 - (-1)] = \frac{\pi}{12}
 \end{aligned}$$