11.3 PARTIAL DERIVATIVES

A Click here for answers.

S Click here for solutions.

1–14 ▪ Find the indicated partial derivatives.

1. $f(x, y) = x^3y^5$; $f_x(3, -1)$

2. $f(x, y) = \sqrt{2x + 3y}$; $f_y(2, 4)$

3. $f(x, y) = xe^{-y} + 3y$; $\dfrac{\partial f}{\partial y}(1, 0)$

4. $f(x, y) = \sin(y - x)$; $\dfrac{\partial f}{\partial y}(3, 3)$

5. $z = \dfrac{x^3 + y^3}{x^2 + y^2}$; $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$

6. $z = x\sqrt{y} - \dfrac{y}{\sqrt{x}}$; $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$

7. $z = \dfrac{x}{y} + \dfrac{y}{x}$; $\dfrac{\partial z}{\partial x}$

8. $z = (3xy^2 - x^4 + 1)^4$; $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$

9. $u = xy\sec(xy)$; $\dfrac{\partial u}{\partial x}$

10. $u = \dfrac{x}{x + t}$; $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial t}$

11. $f(x, y, z) = xyz$; $f_y(0, 1, 2)$

12. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; $f_z(0, 3, 4)$

13. $u = xy + yz + zx$; u_x, u_y, u_z

14. $u = x^2y^3t^4$; u_x, u_y, u_t

15–41 ▪ Find the first partial derivatives of the function.

15. $f(x, y) = x^3y^5 - 2x^2y + x$

16. $f(x, y) = x^2y^2(x^4 + y^4)$

17. $f(x, y) = x^4 + x^2y^2 + y^4$

18. $f(x, y) = \ln(x^2 + y^2)$

19. $f(x, y) = e^x \tan(x - y)$

20. $f(s, t) = s/\sqrt{s^2 + t^2}$

21. $g(x, y) = y\tan(x^2y^3)$

22. $g(x, y) = \ln(x + \ln y)$

23. $f(x, y) = e^{xy}\cos x \sin y$

24. $f(s, t) = \sqrt{2 - 3s^2 - 5t^2}$

25. $z = \sinh\sqrt{3x + 4y}$

26. $z = \log_x y$

27. $f(u, v) = \tan^{-1}(u/v)$

28. $f(x, t) = e^{\sin(t/x)}$

29. $z = \ln\left(x + \sqrt{x^2 + y^2}\right)$

30. $z = x^{x^y}$

31. $f(x, y) = \displaystyle\int_x^y e^{t^2}\,dt$

32. $f(x, y) = \displaystyle\int_y^x \frac{e^t}{t}\,dt$

33. $f(x, y, z) = x^2yz^3 + xy - z$

34. $f(x, y, z) = x\sqrt{yz}$

35. $f(x, y, z) = x^{yz}$

36. $f(x, y, z) = xe^y + ye^z + ze^x$

37. $u = z\sin\dfrac{y}{x + z}$

38. $u = xy^2z^3\ln(x + 2y + 3z)$

39. $u = x^{y^z}$

40. $f(x, y, z, t) = \dfrac{x - y}{z - t}$

41. $f(x, y, z, t) = xy^2z^3t^4$

42–45 ▪ Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

42. $xy + yz = xz$

43. $xyz = \cos(x + y + z)$

44. $x^2 + y^2 - z^2 = 2x(y + z)$

45. $xy^2z^3 + x^3y^2z = x + y + z$

46. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = f(ax + by)$.

47–52 ▪ Find all the second partial derivatives.

47. $f(x, y) = x^2y + x\sqrt{y}$

48. $f(x, y) = \sin(x + y) + \cos(x - y)$

49. $z = (x^2 + y^2)^{3/2}$

50. $z = \cos^2(5x + 2y)$

51. $z = t\sin^{-1}\sqrt{x}$

52. $z = x^{\ln t}$

53–56 ▪ Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$.

53. $u = x^5y^4 - 3x^2y^3 + 2x^2$

54. $u = \sin^2 x \cos y$

55. $u = \sin^{-1}(xy^2)$

56. $u = x^2y^3z^4$

57–63 ▪ Find the indicated partial derivative.

57. $f(x, y) = x^2y^3 - 2x^4y$; f_{xxx}

58. $f(x, y) = e^{xy^2}$; f_{xxy}

59. $f(x, y, z) = x^5 + x^4y^4z^3 + yz^2$; f_{xyz}

60. $f(x, y, z) = e^{xyz}$; f_{yzy}

61. $z = x\sin y$; $\dfrac{\partial^3 z}{\partial y^2\,\partial x}$

62. $z = \ln\sin(x - y)$; $\dfrac{\partial^3 z}{\partial y\,\partial x^2}$

63. $u = \ln(x + 2y^2 + 3z^3)$; $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$

64. If f and g are twice differentiable functions of a single variable, show that the function

$$u(x, y) = xf(x + y) + yg(x + y)$$

satisfies the equation $u_{xx} - 2u_{xy} + u_{yy} = 0$.

65. Show that the function

$$f(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_n^2)^{(2-n)/2}$$

satisfies the equation

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

11.3 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. -27
2. $\frac{3}{8}$
3. 2
4. -1
5. $\dfrac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2}, \dfrac{3x^2y^2+y^4-2yx^3}{(x^2+y^2)^2}$
6. $\sqrt{y}+\dfrac{y}{2x^{3/2}}, \dfrac{x}{2\sqrt{y}}-\dfrac{1}{\sqrt{x}}$
7. $\dfrac{1}{y}-\dfrac{y}{x^2}$
8. $4\left(3xy^2-x^4+1\right)^3\left(3y^2-4x^3\right), 24xy\left(3xy^2-x^4+1\right)^3$
9. $y\sec(xy)\,[1+xy\tan(xy)]$
10. $\dfrac{t}{(x+t)^2}, -\dfrac{x}{(x+t)^2}$
11. 0
12. $\frac{4}{5}$
13. $y+z, x+z, y+x$
14. $2xy^3t^4, 3x^2y^2t^4, 4x^2y^3t^3$
15. $f_x(x,y)=3x^2y^5-4xy+1, f_y(x,y)=5x^3y^4-2x^2$
16. $f_x(x,y)=6x^5y^2+2xy^6, f_y(x,y)=6y^5x^2+2yx^6$
17. $f_x(x,y)=4x^3+2xy^2, f_y(x,y)=2x^2y+4y^3$
18. $f_x(x,y)=\dfrac{2x}{x^2+y^2}, f_y(x,y)=\dfrac{2y}{x^2+y^2}$
19. $f_x(x,y)=e^x\left[\tan(x-y)+\sec^2(x-y)\right]$, $f_y(x,y)=-e^x\sec^2(x-y)$
20. $f_s(s,t)=\dfrac{t^2}{(s^2+t^2)^{3/2}}, f_t(s,t)=-\dfrac{st}{(s^2+t^2)^{3/2}}$
21. $g_x(x,y)=2xy^4\sec^2\left(x^2y^3\right)$, $g_y(x,y)=\tan\left(x^2y^3\right)+3x^2y^3\sec^2\left(x^2y^3\right)$
22. $g_x(x,y)=\dfrac{1}{x+\ln y}, g_y(x,y)=\dfrac{1}{y(x+\ln y)}$
23. $f_x(x,y)=e^{xy}\sin y\,(y\cos x-\sin x)$, $f_y(x,y)=e^{xy}\cos x\,(x\sin y+\cos y)$
24. $f_s(s,t)=-\dfrac{3s}{\sqrt{2-3s^2-5t^2}}$, $f_t(s,t)=-\dfrac{5t}{\sqrt{2-3s^2-5t^2}}$
25. $\dfrac{\partial z}{\partial x}=\dfrac{3\cosh\sqrt{3x+4y}}{2\sqrt{3x+4y}}, \dfrac{\partial z}{\partial y}=\dfrac{2\cosh\sqrt{3x+4y}}{\sqrt{3x+4y}}$
26. $\dfrac{\partial z}{\partial x}=-\dfrac{\ln y}{x(\ln x)^2}, \dfrac{\partial z}{\partial y}=\dfrac{1}{y\ln x}$
27. $f_u(u,v)=\dfrac{v}{u^2+v^2}, f_v(u,v)=-\dfrac{u}{u^2+v^2}$
28. $f_x(x,t)=-t\cos\left(\dfrac{t}{x}\right)\dfrac{e^{\sin(t/x)}}{x^2}$, $f_t(x,t)=\dfrac{e^{\sin(t/x)}}{x}\cos\left(\dfrac{t}{x}\right)$
29. $\dfrac{\partial z}{\partial x}=\dfrac{1}{\sqrt{x^2+y^2}}, \dfrac{\partial z}{\partial y}=\dfrac{y}{x\sqrt{x^2+y^2}+x^2+y^2}$
30. $\dfrac{\partial z}{\partial x}=x^{y-1}x^{x^y}(1+y\ln x), \dfrac{\partial z}{\partial y}=x^{x^y+y}(\ln x)^2$
31. $f_x(x,y)=-e^{x^2}, f_y(x,y)=e^{y^2}$
32. $f_x(x,y)=\dfrac{e^x}{x}, f_y(x,y)=-\dfrac{e^y}{y}$
33. $f_x(x,y,z)=2xyz^3+y, f_y(x,y,z)=x^2z^3+x$, $f_z(x,y,z)=3x^2yz^2-1$
34. $f_x(x,y,z)=\sqrt{yz}, f_y(x,y,z)=\dfrac{xz}{2\sqrt{yz}}$, $f_z(x,y,z)=\dfrac{xy}{2\sqrt{yz}}$
35. $f_x(x,y,z)=yzx^{yz-1}, f_y(x,y,z)=zx^{yz}\ln x$, $f_z(x,y,z)=yx^{yz}\ln x$
36. $f_x(x,y,z)=e^y+ze^x, f_y(x,y,z)=xe^y+e^z$, $f_z(x,y,z)=ye^z+e^x$
37. $u_x=\dfrac{-yz}{(x+z)^2}\cos\left(\dfrac{y}{x+z}\right), u_y=\dfrac{z}{x+z}\cos\left(\dfrac{y}{x+z}\right)$, $u_z=\sin\left(\dfrac{y}{x+z}\right)-\dfrac{yz}{(x+z)^2}\cos\left(\dfrac{y}{x+z}\right)$
38. $u_x=y^2z^3\left[\ln(x+2y+3z)+\dfrac{x}{x+2y+3z}\right]$, $u_y=2xyz^3\left[\ln(x+2y+3z)+\dfrac{y}{x+2y+3z}\right]$, $u_z=3xy^2z^2\left[\ln(x+2y+3z)+\dfrac{z}{x+2y+3z}\right]$
39. $u_x=y^zx^{y^z-1}, u_y=x^{y^z}y^{z-1}z\ln x, u_z=x^{y^z}y^z\ln x\ln y$
40. $f_x(x,y,z,t)=\dfrac{1}{z-t}, f_y(x,y,z,t)=-\dfrac{1}{z-t}$, $f_z(x,y,z,t)=\dfrac{y-x}{(z-t)^2}, f_t(x,y,z,t)=\dfrac{x-y}{(z-t)^2}$
41. $f_x(x,y,z,t)=y^2z^3t^4, f_y(x,y,z,t)=2xyz^3t^4$, $f_z(x,y,z,t)=3xy^2z^2t^4, f_t(x,y,z,t)=4xy^2z^3t^3$
42. $\dfrac{z-y}{y-x}, \dfrac{x+z}{x-y}$
43. $-\dfrac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}, -\dfrac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}$

44. $\dfrac{x-y-z}{x+z}, \dfrac{y-x}{x+z}$

45. $\dfrac{1-y^2z^3-3x^2y^2z}{3xy^2z^2+x^3y^2-1}, \dfrac{1-2xyz^3-2x^3yz}{3xy^2z^2+x^3y^2-1}$

46. $af'(ax+by), bf'(ax+by)$

47. $f_{xx}=2y, f_{xy}=2x+\dfrac{1}{2\sqrt{y}}, f_{yx}=2x+\dfrac{1}{2\sqrt{y}},$
$f_{yy}=-\dfrac{x}{4y^{3/2}}$

48. $f_{xx}=-\sin(x+y)-\cos(x-y),$
$f_{xy}=-\sin(x+y)+\cos(x-y),$
$f_{yx}=-\sin(x+y)+\cos(x-y),$
$f_{yy}=-\sin(x+y)-\cos(x-y)$

49. $z_{xx}=\dfrac{3\left(2x^2+y^2\right)}{\sqrt{x^2+y^2}}, z_{xy}=\dfrac{3xy}{\sqrt{x^2+y^2}}, z_{yx}=\dfrac{3xy}{\sqrt{x^2+y^2}},$
$z_{yy}=\dfrac{3\left(x^2+2y^2\right)}{\sqrt{x^2+y^2}}$

50. $z_{xx}=50\left[\sin^2(5x+2y)-\cos^2(5x+2y)\right],$
$z_{xy}=20\left[\sin^2(5x+2y)-\cos^2(5x+2y)\right],$
$z_{yx}=20\left[\sin^2(5x+2y)-\cos^2(5x+2y)\right],$
$z_{yy}=8\left[\sin^2(5x+2y)-\cos^2(5x+2y)\right]$

51. $z_{xx}=\dfrac{t(2x-1)}{4(x-x^2)^{3/2}}, z_{xt}=\dfrac{1}{2\sqrt{x-x^2}}, z_{tx}=\dfrac{1}{2\sqrt{x-x^2}},$
$z_{tt}=0$

52. $z_{xx}=(\ln t)[(\ln t)-1]x^{(\ln t)-2},$
$z_{xt}=x^{(\ln t)-1}\dfrac{1+\ln t\ln x}{t}, z_{tx}=x^{(\ln t)-1}\dfrac{1+\ln t\ln x}{t},$
$z_{tt}=x^{\ln t}\ln x\dfrac{(\ln x)-1}{t^2}$

57. $-48xy$

58. $2y^3e^{xy^2}\left(2+xy^2\right)$

59. $f_{xyz}=48x^3y^3z^2$

60. $x^2z(2+xyz)e^{xyz}$

61. $-\sin y$

62. $-2\csc^2(x-y)\cot(x-y)$

63. $\dfrac{72yz^2}{(x+2y^2+3z^3)^3}$

11.3 SOLUTIONS

E Click here for exercises.

1. $f(x,y) = x^3y^5 \Rightarrow f_x(x,y) = 3x^2y^5$, $f_x(3,-1) = -27$

2. $f(x,y) = \sqrt{2x+3y} \Rightarrow$ $f_y(x,y) = \frac{1}{2}(2x+3y)^{-1/2}(3)$, $f_y(2,4) = \dfrac{3/2}{\sqrt{4+12}} = \dfrac{3}{8}$

3. $f(x,y) = xe^{-y} + 3y \Rightarrow \partial f/\partial y = x(-1)e^{-y} + 3$, $(\partial f/\partial y)(1,0) = -1 + 3 = 2$

4. $f(x,y) = \sin(y-x) \Rightarrow \partial f/\partial x = -\cos(y-x)$, $(\partial f/\partial x)(3,3) = -\cos(0) = -1$

5. $z = \dfrac{x^3+y^3}{x^2+y^2} \Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{3x^2(x^2+y^2) - (x^3+y^3)(2x)}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2},$$

$$\frac{\partial z}{\partial y} = \frac{3y^2(x^2+y^2) - (x^3+y^3)(2y)}{(x^2+y^2)^2} = \frac{3x^2y^2+y^4-2yx^3}{(x^2+y^2)^2}$$

6. $z = x\sqrt{y} - \dfrac{y}{\sqrt{x}} \Rightarrow$

$$\frac{\partial z}{\partial x} = \sqrt{y} - y\left(-\tfrac{1}{2}\right)x^{-3/2} = \sqrt{y} + \frac{y}{2x^{3/2}},$$

$$\frac{\partial z}{\partial y} = x\left(\tfrac{1}{2}\right)y^{-1/2} - \frac{1}{\sqrt{x}} = \frac{x}{2\sqrt{y}} - \frac{1}{\sqrt{x}}$$

7. $z = \dfrac{x}{y} + \dfrac{y}{x} \Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{1}{y} - \dfrac{y}{x^2}$

8. $z = (3xy^2 - x^4 + 1)^4 \Rightarrow$

$$\partial z/\partial x = 4(3xy^2 - x^4 + 1)^3(3y^2 - 4x^3),$$

$$\partial z/\partial y = 4(3xy^2 - x^4 + 1)^3(6xy) = 24xy(3xy^2 - x^4 + 1)^3$$

9. $u = xy\sec(xy) \Rightarrow$

$$\partial u/\partial x = y\sec(xy) + xy[\sec(xy)\tan(xy)](y) = y\sec(xy)[1 + xy\tan(xy)]$$

10. $u = \dfrac{u}{x+t} \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{1(x+t) - x(1)}{(x+t)^2} = \dfrac{t}{(x+t)^2}$,

$$\frac{\partial u}{\partial t} = x(-1)(x+t)^{-2}(1) = -\frac{x}{(x+t)^2}$$

11. $f(x,y,z) = xyz \Rightarrow f_y(x,y,z) = xz$, so $f_y(0,1,2) = 0$.

12. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \Rightarrow$ $f_z(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)$, so $f_z(0,3,4) = \frac{4}{\sqrt{0+9+16}} = \frac{4}{5}$.

13. $u = xy + yz + zx \Rightarrow u_x = y + z$, $u_y = x + z$, $u_z = y + x$

14. $u = x^2y^3t^4 \Rightarrow u_x = 2xy^3t^4$, $u_y = 3x^2y^2t^4$, $u_t = 4x^2y^3t^3$

15. $f(x,y) = x^3y^5 - 2x^2y + x \Rightarrow$ $f_x(x,y) = 3x^2y^5 - 4xy + 1$, $f_y(x,y) = 5x^3y^4 - 2x^2$

16. $f(x,y) = x^2y^2(x^4+y^4) \Rightarrow$ $f_x(x,y) = 2xy^2(x^4+y^4) + x^2y^2(4x^3) = 6x^5y^2 + 2xy^6$ and by symmetry $f_y(x,y) = 6y^5x^2 + 2yx^6$.

17. $f(x,y) = x^4 + x^2y^2 + y^4 \Rightarrow f_x(x,y) = 4x^3 + 2xy^2$, $f_y(x,y) = 2x^2y + 4y^3$

18. $f(x,y) = \ln(x^2+y^2) \Rightarrow$

$$f_x(x,y) = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2},\ f_y(x,y) = \frac{2y}{x^2+y^2}$$

19. $f(x,y) = e^x\tan(x-y) \Rightarrow$

$$f_x(x,y) = e^x\tan(x-y) + e^x\sec^2(x-y) = e^x[\tan(x-y) + \sec^2(x-y)],$$

$$f_y(x,y) = e^x[\sec^2(x-y)](-1) = -e^x\sec^2(x-y)$$

20. $f(s,t) = \dfrac{s}{\sqrt{s^2+t^2}} \Rightarrow$

$$f_s(s,t) = \frac{(1)\sqrt{s^2+t^2} - s\left(\frac{1}{2}\right)(s^2+t^2)^{-1/2}(2s)}{\left(\sqrt{s^2+t^2}\right)^2} = \frac{|s^2+t^2| - s^2}{|s^2+t^2|\sqrt{s^2+t^2}} = \frac{t^2}{(s^2+t^2)^{3/2}},$$

$$f_t(s,t) = s\left(-\tfrac{1}{2}\right)(s^2+t^2)^{-3/2}(2t) = -\frac{st}{(s^2+t^2)^{3/2}}$$

21. $g(x,y) = y\tan(x^2y^3) \Rightarrow$

$$g_x(x,y) = [y\sec^2(x^2y^3)](2xy^3) = 2xy^4\sec^2(x^2y^3),$$

$$g_y(x,y) = \tan(x^2y^3) + [y\sec^2(x^2y^3)](3x^2y^2) = \tan(x^2y^3) + 3x^2y^3\sec^2(x^2y^3)$$

22. $g(x,y) = \ln(x + \ln y) \Rightarrow$

$$g_x(x,y) = \frac{1}{x+\ln y}(1) = \frac{1}{x+\ln y},$$

$$g_y(x,y) = \frac{1}{x+\ln y}\left(\frac{1}{y}\right) = \frac{1}{y(x+\ln y)}$$

23. $f(x,y) = e^{xy}\cos x\sin y \Rightarrow$

$$f_x(x,y) = ye^{xy}\cos x\sin y + e^{xy}(-\sin x)\sin y = e^{xy}\sin y(y\cos x - \sin x),$$

$$f_y(x,y) = xe^{xy}\cos x\sin y + e^{xy}\cos x\cos y = e^{xy}\cos x(x\sin y + \cos y)$$

24. $f(s,t) = \sqrt{2-3s^2-5t^2} \quad \Rightarrow$

$$f_s(s,t) = \tfrac{1}{2}\left(2-3s^2-5t^2\right)^{-1/2}(-6s) = -\frac{3s}{\sqrt{2-3s^2-5t^2}},$$

$$f_t(s,t) = \tfrac{1}{2}\left(2-3s^2-5t^2\right)^{-1}(-10t) = -\frac{5t}{\sqrt{2-3s^2-5t^2}}$$

25. $z = \sinh\sqrt{3x+4y} \quad \Rightarrow$

$$\frac{\partial z}{\partial x} = \left(\cosh\sqrt{3x+4y}\right)\left(\tfrac{1}{2}\right)(3x+4y)^{-1/2}(3) = \frac{3\cosh\sqrt{3x+4y}}{2\sqrt{3x+4y}},$$

$$\frac{\partial z}{\partial y} = \left(\cosh\sqrt{3x+4y}\right)\left(\tfrac{1}{2}\right)(3x+4y)^{-1/2}(4) = \frac{2\cosh\sqrt{3x+4y}}{\sqrt{3x+4y}}$$

26. Since $z = \log_x y$, $x^z = y$ and $z\ln x = \ln y$. Then

$$\frac{\partial z}{\partial x}\ln x + z\left(\frac{1}{x}\right) = 0, \text{ so } \frac{\partial z}{\partial x} = -\frac{z}{x\ln x} = -\frac{\ln y}{x(\ln x)^2}.$$

Also, $(\ln x)\dfrac{\partial z}{\partial y} = \dfrac{1}{y}$, so $\dfrac{\partial z}{\partial y} = \dfrac{1}{y\ln x}$.

27. $f(u,v) = \tan^{-1}\left(\dfrac{u}{v}\right) \quad \Rightarrow$

$$f_u(u,v) = \frac{1}{1+(u/v)^2}\left(\frac{1}{v}\right) = \frac{1}{v}\left(\frac{v^2}{u^2+v^2}\right) = \frac{v}{u^2+v^2},$$

$$f_v(u,v) = \frac{1}{1+(u/v)^2}\left(-\frac{u}{v^2}\right) = -\frac{u}{v^2}\left(\frac{v^2}{u^2+v^2}\right) = -\frac{u}{u^2+v^2}$$

28. $f(x,t) = e^{\sin(t/x)} \quad \Rightarrow$

$$f_x(x,t) = e^{\sin(t/x)}\cos\left(\frac{t}{x}\right)\left(-\frac{t}{x^2}\right) = -t\cos\left(\frac{t}{x}\right)\frac{e^{\sin(t/x)}}{x^2},$$

$$f_t(x,t) = e^{\sin(t/x)}\cos\left(\frac{t}{x}\right)\left(\frac{1}{x}\right) = \frac{e^{\sin(t/x)}}{x}\cos\left(\frac{t}{x}\right)$$

29. $z = \ln\left(x+\sqrt{x^2+y^2}\right) \quad \Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{1}{x+\sqrt{x^2+y^2}}\left[1+\tfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x)\right] = \frac{\left(\sqrt{x^2+y^2}+x\right)\Big/\sqrt{x^2+y^2}}{\left(x+\sqrt{x^2+y^2}\right)} = \frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+\sqrt{x^2+y^2}}\left(\frac{1}{2}\right)\left(x^2+y^2\right)^{-1/2}(2y) = \frac{y}{x\sqrt{x^2+y^2}+x^2+y^2}$$

30. $z = x^{x^y}$, so $\ln z = x^y\ln x$ and

$$\frac{1}{z}\frac{\partial z}{\partial x} = yx^{y-1}\ln x + x^y\left(\frac{1}{x}\right) \quad \Leftrightarrow$$

$$\frac{\partial z}{\partial x} = z\left[yx^{y-1}\ln x + x^{y-1}\right] = x^{y-1}x^{x^y}(1+y\ln x),$$

$$\frac{\partial z}{\partial y} = \left(x^{x^y}\right)(\ln x)\frac{\partial}{\partial y}(x^y) = \left(x^{x^y}\right)(\ln x)\,x^y\ln x = x^{x^y+y}(\ln x)^2$$

31. $f(x,y) = \displaystyle\int_x^y e^{t^2}\,dt$. By FTC1,

$\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ for f continuous. Thus

$$f_x(x,y) = \frac{\partial}{\partial x}\int_x^y e^{t^2}\,dt = \frac{\partial}{\partial x}\left(-\int_y^x e^{t^2}\,dt\right) = -e^{x^2}$$

and $f_y(x,y) = \dfrac{\partial}{\partial y}\displaystyle\int_x^y e^{t^2}\,dt = e^{y^2}$.

32. $f(x,y) = \displaystyle\int_y^x \frac{e^t}{t}\,dt$. If 0 isn't in the interval $[y,x]$, then by FTC1, $f_x(x,y) = \dfrac{e^x}{x}$ and $f_y(x,y) = -\dfrac{e^y}{y}$.

33. $f(x,y,z) = x^2yz^3 + xy - z \quad \Rightarrow$

$f_x(x,y,z) = 2xyz^3 + y$, $f_y(x,y,z) = x^2z^3 + x$,

$f_z(x,y,z) = 3x^2yz^2 - 1$

34. $f(x,y,z) = x\sqrt{yz} \quad \Rightarrow \quad f_x(x,y,z) = \sqrt{yz}$,

$f_y(x,y,z) = x\left(\tfrac{1}{2}\right)(yz)^{-1/2}(z) = \dfrac{xz}{2\sqrt{yz}}$, and by symmetry, $f_z(x,y,z) = \dfrac{xy}{2\sqrt{yz}}$.

35. $f(x,y,z) = x^{yz} \quad \Rightarrow \quad f_x(x,y,z) = yzx^{yz-1}$. By Theorem 3.3.6, $f_y(x,y,z) = x^{yz}\ln(x^z) = zx^{yz}\ln x$ and by symmetry $f_z(x,y,z) = yx^{yz}\ln x$.

36. $f(x,y,z) = xe^y + ye^z + ze^x \quad \Rightarrow$

$f_x(x,y,z) = e^y + ze^x$, $f_y(x,y,z) = xe^y + e^z$,

$f_z(x,y,z) = ye^z + e^x$

37. $u = z\sin\left(\dfrac{y}{x+z}\right) \quad \Rightarrow$

$$u_x = z\cos\left(\frac{y}{x+z}\right)\left[-y(x+z)^{-2}\right] = \frac{-yz}{(x+z)^2}\cos\left(\frac{y}{x+z}\right),$$

$$u_y = z\cos\left(\frac{y}{x+z}\right)\left(\frac{1}{x+z}\right) = \frac{z}{x+z}\cos\left(\frac{y}{x+z}\right),$$

$$u_z = \sin\left(\frac{y}{x+z}\right) + z\cos\left(\frac{y}{x+z}\right)\left[-y(x+z)^{-2}\right] = \sin\left(\frac{y}{x+z}\right) - \frac{yz}{(x+z)^2}\cos\left(\frac{y}{x+z}\right)$$

38. $u = xy^2z^3 \ln(x + 2y + 3z) \;\; \Rightarrow$

$$u_x = y^2z^3 \ln(x + 2y + 3z) = xy^2z^3 \left(\frac{1}{x + 2y + 3z}\right)$$
$$= y^2z^3 \left[\ln(x + 2y + 3z) + \frac{x}{x + 2y + 3z}\right],$$
$$u_y = 2xyz^3 \ln(x + 2y + 3z) + xy^2z^3 \left(\frac{1}{x + 2y + 3z}\right)(2)$$
$$= 2xyz^3 \left[\ln(x + 2y + 3z) + \frac{y}{x + 2y + 3z}\right],$$

and by symmetry,

$$u_z = 3xy^2z^2 \left[\ln(x + 2y + 3z) + \frac{z}{x + 2y + 3z}\right].$$

39. $u = x^{y^z} \;\; \Rightarrow \;\; u_x = y^z x^{y^z - 1}$,

$u_y = x^{y^z} \ln x \cdot zy^{z-1} = x^{y^z} y^{z-1} z \ln x$,

$u_z = x^{y^z} \ln x \,(y^z \ln y) = x^{y^z} y^z \ln x \ln y$

40. $f(x, y, z, t) = \dfrac{x - y}{z - t} \;\; \Rightarrow \;\; f_x(x, y, z, t) = \dfrac{1}{z - t}$,

$f_y(x, y, z, t) = -\dfrac{1}{z - t}$,

$f_z(x, y, z, t) = (x - y)(-1)(z - t)^{-2} = \dfrac{y - x}{(z - t)^2}$, and

$f_t(x, y, z, t) = (x - y)(-1)(z - t)^{-2}(-1) = \dfrac{x - y}{(z - t)^2}$.

41. $f(x, y, z, t) = xy^2z^3t^4 \;\; \Rightarrow \;\; f_x(x, y, z, t) = y^2z^3t^4$,

$f_y(x, y, z, t) = 2xyz^3t^4$, $f_z(x, y, z, t) = 3xy^2z^2t^4$, and

$f_t(x, y, z, t) = 4xy^2z^3t^3$.

42. $xy + yz = xz \;\; \Rightarrow \;\; \dfrac{\partial}{\partial x}(xy + yz) = \dfrac{\partial}{\partial x}(xz) \;\; \Leftrightarrow$

$y + y\dfrac{\partial z}{\partial x} = z + x\dfrac{\partial z}{\partial x} \;\; \Leftrightarrow \;\; (y - x)\dfrac{\partial z}{\partial x} = z - y$, so

$\dfrac{\partial z}{\partial x} = \dfrac{z - y}{y - x}$. $\dfrac{\partial}{\partial y}(xy + yz) = \dfrac{\partial}{\partial y}(xz) \;\; \Leftrightarrow$

$x + z + y\dfrac{\partial z}{\partial y} = x\dfrac{\partial z}{\partial y} \;\; \Leftrightarrow \;\; (y - x)\dfrac{\partial z}{\partial y} = -(x + z)$, so

$\dfrac{\partial z}{\partial y} = \dfrac{x + z}{x - y}$.

43. $xyz = \cos(x + y + z) \;\; \Rightarrow$

$\dfrac{\partial}{\partial x}(xyz) = \dfrac{\partial}{\partial x}[\cos(x + y + z)] \;\; \Leftrightarrow$

$yz + xy\dfrac{\partial z}{\partial x} = [-\sin(x + y + z)]\left(1 + \dfrac{\partial z}{\partial x}\right)$,

$[xy + \sin(x + y + z)]\dfrac{\partial z}{\partial x} = -[yz + \sin(x + y + z)]$,

so $\dfrac{\partial z}{\partial x} = -\dfrac{yz + \sin(x + y + z)}{xy + \sin(x + y + z)}$,

$\dfrac{\partial}{\partial y}(xyz) = \dfrac{\partial}{\partial y}(\cos(x + y + z))$, and so by symmetry,

$\dfrac{\partial z}{\partial y} = -\dfrac{xz + \sin(x + y + z)}{xy + \sin(x + y + z)}$.

44. $x^2 + y^2 - z^2 = 2x(y + z) \;\; \Leftrightarrow$

$\dfrac{\partial}{\partial x}(x^2 + y^2 - z^2) = \dfrac{\partial}{\partial x}[2x(y + z)] \;\; \Leftrightarrow$

$2x - 2z\dfrac{\partial z}{\partial x} = 2(y + z) + 2x\dfrac{\partial z}{\partial x} \;\; \Leftrightarrow$

$2(x + z)\dfrac{\partial z}{\partial x} = 2(x - y - z)$, so $\dfrac{\partial z}{\partial x} = \dfrac{x - y - z}{x + z}$.

$\dfrac{\partial}{\partial y}(x^2 + y^2 - z^2) = \dfrac{\partial}{\partial y}[2x(y + z)] \;\; \Leftrightarrow$

$2y - 2z\dfrac{\partial z}{\partial y} = 2x\left(1 + \dfrac{\partial z}{\partial y}\right) \;\; \Leftrightarrow$

$2(x + z)\dfrac{\partial z}{\partial y} = 2(y - x)$, so $\dfrac{\partial z}{\partial y} = \dfrac{y - x}{x + z}$.

45. $xy^2z^3 + x^3y^2z = x + y + z \;\; \Rightarrow$

$\dfrac{\partial}{\partial x}(xy^2z^3 + x^3y^2z) = \dfrac{\partial}{\partial x}(x + y + z) \;\; \Leftrightarrow$

$y^2z^3 + 3xy^2z^2\dfrac{\partial z}{\partial x} + 3x^2y^2z + x^3y^2\dfrac{\partial z}{\partial x} = 1 + \dfrac{\partial z}{\partial x}$, so

$(3xy^2z^2 + x^3y^2 - 1)\dfrac{\partial z}{\partial x} = 1 - y^2z^3 - 3x^2y^2z$ and

$\dfrac{\partial z}{\partial x} = \dfrac{1 - y^2z^3 - 3x^2y^2z}{3xy^2z^2 + x^3y^2 - 1}$.

$\dfrac{\partial}{\partial y}(xy^2z^3 + x^3y^2z) = \dfrac{\partial}{\partial y}(x + y + z) \;\; \Leftrightarrow$

$2xyz^3 + 3xy^2z^2\dfrac{\partial z}{\partial y} + 2x^3yz + x^3y^2\dfrac{\partial z}{\partial y} = 1 + \dfrac{\partial z}{\partial y}$, so

$(3xy^2z^2 + x^3y^2 - 1)\dfrac{\partial z}{\partial y} = 1 - 2xyz^3 - 2x^3yz$ and

$\dfrac{\partial z}{\partial y} = \dfrac{1 - 2xyz^3 - 2x^3yz}{3xy^2z^2 + x^3y^2 - 1}$.

46. $z = f(ax + by)$. Let $u = ax + by$.

Then $\dfrac{\partial u}{\partial x} = a$ and $\dfrac{\partial u}{\partial y} = b$. Hence

$\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = a\dfrac{df}{d(ax + by)} = af'(ax + by)$ and

$\dfrac{\partial z}{\partial y} = b\dfrac{df}{d(ax + by)} = bf'(ax + by)$.

47. $f(x, y) = x^2y + x\sqrt{y} \;\; \Rightarrow \;\; f_x = 2xy + \sqrt{y}$,

$f_y = x^2 + \dfrac{x}{2\sqrt{y}}$. Thus $f_{xx} = 2y$, $f_{xy} = 2x + \dfrac{1}{2\sqrt{y}}$,

$f_{yx} = 2x + \dfrac{1}{2\sqrt{y}}$ and $f_{yy} = -\dfrac{x}{4y^{3/2}}$.

48. $f(x, y) = \sin(x + y) + \cos(x - y) \;\; \Rightarrow$

$f_x = \cos(x + y) - \sin(x - y)$,

$f_y = \cos(x + y) + \sin(x - y)$. Thus

$f_{xx} = -\sin(x + y) - \cos(x - y)$,

$f_{xy} = -\sin(x + y) + \cos(x - y)$,

$f_{yx} = -\sin(x + y) + \cos(x - y)$ and

$f_{yy} = -\sin(x + y) - \cos(x - y)$.

49. $z = \left(x^2+y^2\right)^{3/2} \Rightarrow$

$z_x = \frac{3}{2}\left(x^2+y^2\right)^{1/2}(2x) = 3x\left(x^2+y^2\right)^{1/2}$ and

$z_y = 3y\left(x^2+y^2\right)^{1/2}$. Thus

$$z_{xx} = 3\left(x^2+y^2\right)^{1/2} + 3x\left(x^2+y^2\right)^{-1/2}\left(\tfrac{1}{2}\right)(2x) = \frac{3\left(x^2+y^2\right)+3x^2}{\sqrt{x^2+y^2}} = \frac{3\left(2x^2+y^2\right)}{\sqrt{x^2+y^2}}$$ and

$z_{xy} = 3x\left(\frac{1}{2}\right)\left(x^2+y^2\right)^{-1/2}(2y) = \dfrac{3xy}{\sqrt{x^2+y^2}}$. By

symmetry $z_{yx} = \dfrac{3xy}{\sqrt{x^2+y^2}}$ and $z_{yy} = \dfrac{3\left(x^2+2y^2\right)}{\sqrt{x^2+y^2}}$.

50. $z = \cos^2(5x+2y) \Rightarrow$

$$z_x = [2\cos(5x+2y)][-\sin(5x+2y)](5) = -10\cos(5x+2y)\sin(5x+2y)$$ and

$$z_y = [2\cos(5x+2y)][-\sin(5x+2y)](2) = -4\cos(5x+2y)\sin(5x+2y)$$

Thus

$$z_{xx} = (10)(5)\sin^2(5x+2y) + (-10)(5)\cos^2(5x+2y) = 50\left[\sin^2(5x+2y) - \cos^2(5x+2y)\right],$$

$$z_{xy} = (10)(2)\sin^2(5x+2y) + (-10)(2)\cos^2(5x+2y) = 20\left[\sin^2(5x+2y) - \cos^2(5x+2y)\right],$$

$$z_{yx} = -(-4)(5)\sin^2(5x+2y) + (-4)(5)\cos^2(5x+2y) = 20\left[\sin^2(5x+2y) - \cos^2(5x+2y)\right],$$

$$z_{yy} = -(-4)(2)\sin^2(5x+2y) + (-4)(2)\cos^2(5x+2y) = 8\left[\sin^2(5x+2y) - \cos^2(5x+2y)\right]$$

51. $z = t\sin^{-1}\sqrt{x} \Rightarrow$

$z_x = t\frac{1}{\sqrt{1-(\sqrt{x})^2}}\left(\frac{1}{2}\right)x^{-1/2} = \dfrac{t}{2\sqrt{x-x^2}}$,

$z_t = \sin^{-1}\sqrt{x}$. Thus

$z_{xx} = \frac{1}{2}t\left(-\frac{1}{2}\right)\left(x-x^2\right)^{-3/2}(1-2x) = \dfrac{t(2x-1)}{4\left(x-x^2\right)^{3/2}}$,

$z_{xt} = \dfrac{1}{2\sqrt{x-x^2}}$,

$z_{tx} = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x-x^2}}$, and $z_{tt} = 0$.

52. $z = x^{\ln t} \Rightarrow z_x = (\ln t)\,x^{(\ln t)-1}$, $\ln z = (\ln t)(\ln x)$ so

$z_t = \left(x^{\ln t}\right)\left(\dfrac{1}{t}\right)\ln x = x^{\ln t}\dfrac{\ln x}{t}$. Thus

$z_{xx} = (\ln t)\left[(\ln t)-1\right]x^{(\ln t)-2}$,

$\ln z_x = \ln(\ln t) + [(\ln t)-1]\ln x$, so

$$z_{xt} = z_x\left[\frac{1}{\ln t}\left(\frac{1}{t}\right) + \frac{1}{t}\ln x\right] = (\ln t)\,x^{(\ln t)-1}\frac{1+(\ln t)(\ln x)}{t\ln t} = x^{(\ln t)-1}\frac{1+\ln t\ln x}{t}$$

Also

$$z_{tx} = \frac{(\ln t)\,x^{(\ln t)-1}\ln x + (1/x)\,x^{\ln t}}{t} = x^{(\ln t)-1}\frac{1+\ln t\ln x}{t}$$ and

$$z_{tt} = \left[\frac{\partial}{\partial t}\left(x^{\ln t}\right)\right]\left(\frac{\ln t}{t}\right) + x^{\ln t}\left[-(\ln x)\,t^{-2}\right] = x^{\ln t}\ln x\frac{(\ln x)-1}{t^2}$$

53. $u = x^5y^4 - 3x^2y^3 + 2x^2 \Rightarrow u_x = 5x^4y^4 - 6xy^3 + 4x$,
$u_{xy} = 20x^4y^3 - 18xy^2$ and $u_y = 4x^5y^3 - 9x^2y^2$,
$u_{yx} = 20x^4y^3 - 18xy^2$. Thus $u_{xy} = u_{yx}$.

54. $u = \sin^2 x\cos y \Rightarrow u_x = 2\sin x\cos x\cos y$,
$u_{xy} = -2\sin x\cos x\sin y$ and $u_y = -\sin^2 x\sin y$,
$u_{yx} = -2\sin x\cos x\sin y$. Thus $u_{xy} = u_{yx}$.

55. $u = \sin^{-1}\left(xy^2\right) \Rightarrow$

$u_x = \dfrac{1}{\sqrt{1-(xy^2)^2}}\left(y^2\right) = y^2\sqrt{1-x^2y^4}$,

$$u_{xy} = 2y\left(1-x^2y^4\right)^{-1/2} + y^2\left(-\tfrac{1}{2}\right)\left(1-x^2y^4\right)^{-3/2}\left(-4x^2y^3\right) = \frac{2y\left(1-x^2y^4\right)+2x^2y^5}{\left(1-x^2y^4\right)^{3/2}} = \frac{2y}{\left(1-x^2y^4\right)^{3/2}}$$ and

$u_y = \dfrac{1}{\sqrt{1-(xy^2)^2}}(2xy) = \dfrac{2xy}{\sqrt{1-x^2y^4}}$,

$$u_{yx} = \frac{2y\sqrt{1-x^2y^4} - 2xy\left(\frac{1}{2}\right)\left(1-x^2y^4\right)^{-1/2}\left(-2xy^4\right)}{1-x^2y^4} = \frac{2y-2x^2y^5+2x^2y^5}{\left(1-x^2y^4\right)^{3/2}} = \frac{2y}{\left(1-x^2y^4\right)^{3/2}}$$

Thus $u_{xy} = u_{yx}$.

56. $u = x^2y^3z^4 \Rightarrow u_x = 2xy^3z^4$, $u_{xy} = 6xy^2z^4$,
$u_{xz} = 8xy^3z^3$; $u_y = 3x^2y^2z^4$, $u_{yx} = 6xy^2z^4$,
$u_{yz} = 12x^2y^2z^3$; $u_z = 4x^2y^3z^3$, $u_{zx} = 8xy^3z^3$,
$u_{zy} = 12x^2y^2z^3$. Thus $u_{xy} = u_{yx}$, $u_{xz} = u_{zy}$, and
$u_{yz} = u_{zy}$.

57. $f(x,y) = x^2y^3 - 2x^4y \Rightarrow f_x = 2xy^3 - 8x^3y$,
$f_{xx} = 2y^3 - 24x^2y$, $f_{xxx} = -48xy$

58. $f(x,y) = e^{xy^2} \Rightarrow f_x = y^2e^{xy^2}$, $f_{xx} = y^4e^{xy^2}$,
$f_{xxy} = 4y^3e^{xy^2} + 2xy^5e^{xy^2} = 2y^3e^{xy^2}\left(2+xy^2\right)$

59. $f(x,y,z) = x^5 + x^4y^4z^3 + yz^2 \Rightarrow$
$f_x = 5x^4 + 4x^3y^4z^3$, $f_{xy} = 16x^3y^3z^3$, and
$f_{xyz} = 48x^3y^3z^2$

60. $f(x,y,z) = e^{xyz} \quad \Rightarrow \quad f_y = xze^{xyz}$,

$f_{yz} = xe^{xyz} + xz(xy)e^{xyz} = xe^{xyz}(1 + yxz)$, and

$$f_{yzy} = x(xz)e^{xyz}(1 + xyz) + xe^{xyz}(xz) = x^2z(2 + xyz)e^{xyz}$$

61. $z = x\sin y \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \sin y$, $\dfrac{\partial^2 z}{\partial y\,\partial x} = \cos y$, and

$\dfrac{\partial^3 z}{\partial y^2\,\partial x} = -\sin y$.

62. $z = \ln\sin(x - y) \quad \Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{1}{\sin(x-y)}\cos(x-y) = \cot(x-y),$$

$$\frac{\partial^2 z}{\partial x^2} = -\csc^2(x-y) \text{ and}$$

$$\frac{\partial^3 z}{\partial y\,\partial x^2} = -2\csc(x-y)\left[-\csc(x-y)\cot(x-y)(-1)\right] = -2\csc^2(x-y)\cot(x-y)$$

63. $u = \ln\left(x + 2y^2 + 3z^3\right) \quad \Rightarrow$

$$\frac{\partial u}{\partial z} = \frac{1}{x + 2y^2 + 3z^3}\left(9z^2\right) = \frac{9z^2}{x + 2y^2 + 3z^3},$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = -9z^2\left(x + 2y^2 + 3z^3\right)^{-2}(4y) = -\frac{36yz^2}{\left(x + 2y^2 + 3z^3\right)^2},$$

and $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \dfrac{72yz^2}{\left(x + 2y^2 + 3z^3\right)^3}$.

64. Let $w = x + y$. Then $\dfrac{\partial w}{\partial x} = 1 = \dfrac{\partial w}{\partial y}$, and by the Chain Rule,

$$u_x = f(w) + x\frac{df}{dw}\frac{\partial w}{\partial x} + y\frac{dg}{dw}\frac{\partial w}{\partial x} = f(w) + xf'(w) + yg'(w),$$

$$u_{xx} = \frac{df}{dw}\frac{\partial w}{\partial x} + f'(w) + x\frac{d\left[f'(w)\right]}{dw}\frac{\partial w}{\partial x} + y\frac{d\left[g'(w)\right]}{dw}\frac{\partial w}{\partial x} = 2f'(w) + xf''(w) + yg''(w),$$

and

$$u_{xy} = \frac{df}{dw}\frac{\partial w}{\partial y} + x\frac{d\left[f'(w)\right]}{dw}\frac{\partial w}{\partial y} + g'(w) + y\frac{d\left[g'(w)\right]}{dw}\frac{\partial w}{\partial y} = f'(w) + xf''(w) + g'(w) + yg''(w)$$

Similarly $u_y = xf'(w) + g(w) + yg'(w)$ and $u_{yy} = xf''(w) + 2g'(w) + yg''(w)$. Then

$$u_{xx} - 2u_{xy} + u_{yy} = 2f'(w) + xf''(w) + yg''(w) - 2f'(w) - 2xf''(w) - 2g'(w) - 2yg''(w) + xf''(w) + 2g'(w) + yg''(w) = 0$$

65. $f(x_1, \ldots, x_n) = \left(x_1^2 + \cdots + x_n^2\right)^{(2-n)/2} \quad \Rightarrow$

$$\frac{\partial f}{\partial x_i} = \left(1 - \frac{n}{2}\right)2x_i\left(x_1^2 + \cdots + x_n^2\right)^{-n/2},\ 1 \le i \le n \quad \Rightarrow$$

$$\frac{\partial^2 f}{\partial x_i^2} = 2\left(1 - \frac{n}{2}\right)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - (2n)\left(1 - \frac{n}{2}\right)\left(x_i^2\right)\left(x_1^2 + \cdots + x_n^2\right)^{-(2+n)/2},$$

$1 \le i \le n$. Therefore

$$\begin{aligned}\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} &= \sum_{i=1}^{n}\left[(2-n)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - n(2-n)\left(x_i^2\right)\left(x_1^2 + \cdots + x_n^2\right)^{-(2+n)/2}\right] \\ &= n(2-n)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - n(2-n)\left(x_1^2 + \cdots + x_n^2\right)\left(x_1^2 + \cdots + x_n^2\right)^{-(2+n)/2} \\ &= n(2-n)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - n(2-n)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} \\ &= 0\end{aligned}$$