### A CATALOG OF ESSENTIAL FUNCTIONS

# A Click here for answers.

1.2

**1–2** • Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)



**3–18** Graph each function, not by plotting points, but by starting with the graph of one of the standard functions given in Section 1.2, and then applying the appropriate transformations.

<b>3.</b> $y = -1/x$	<b>4.</b> $y = 2 - \cos x$
<b>5.</b> $y = \tan 2x$	<b>6.</b> $y = \sqrt[3]{x+2}$
<b>7.</b> $y = \cos(x/2)$	<b>8.</b> $y = x^2 + 2x + 3$
<b>9.</b> $y = \frac{1}{x-3}$	<b>10.</b> $y = -2 \sin \pi x$
$II. \ y = \frac{1}{3}\sin\left(x - \frac{\pi}{6}\right)$	<b>12.</b> $y = 2 + \frac{1}{x+1}$
<b>13.</b> $y = 1 + 2x - x^2$	<b>14.</b> $y = \frac{1}{2}\sqrt{x+4} - 3$
<b>15.</b> $y = 2 - \sqrt{x+1}$	<b>16.</b> $y = (x - 1)^3 + 2$
<b>17.</b> $y =   x  - 1 $	<b>18.</b> $y =  \cos x $

**19–25** Find the functions  $f \circ g$ ,  $g \circ f$ ,  $f \circ f$ , and  $g \circ g$  and their domains.

19. 
$$f(x) = \sqrt{x-1}, \quad g(x) = x^2$$

## S Click here for solutions.

20.	$f(x) = 1/x,  g(x) = x^3 + 2x$
21. ე	$f(x) = \frac{1}{x-1},  g(x) = \frac{x-1}{x+1}$
22.	$f(x) = \sqrt{x^2 - 1},  g(x) = \sqrt{1 - x}$
23.	$f(x) = \sqrt[3]{x},  g(x) = 1 - \sqrt{x}$
24.	$f(x) = \frac{x+2}{2x+1},  g(x) = \frac{x}{x-2}$
25.	$f(x) = \frac{1}{\sqrt{x}}$ $g(x) = x^2 - 4x$
•	
26–2	<b>9</b> Find $f \circ g \circ h$ .
26.	$f(x) = x - 1$ , $g(x) = \sqrt{x}$ , $h(x) = x - 1$
27.	$f(x) = \frac{1}{x},  g(x) = x^3,  h(x) = x^2 + 2$
<b>28</b> . ၂	$f(x) = x^4 + 1$ , $g(x) = x - 5$ , $h(x) = \sqrt{x}$
29.	$f(x) = \sqrt{x},  g(x) = \frac{x}{x-1},  h(x) = \sqrt[3]{x}$
•	
30-3	<b>31</b> • Express the function in the form $f \circ g$ .

- **32.** Suppose we are given the graphs of *f* and *g*, as in the figure, and we want to find the point on the graph of  $h = f \circ g$  that corresponds to x = a. We start at the point (a, 0) and draw a vertical line that intersects the graph of *g* at the point *P*. Then we draw a horizontal line from *P* to the point *Q* on the line y = x.
  - (a) What are the coordinates of P and of Q?
  - (b) If we now draw a vertical line from *Q* to the point *R* on the graph of *f*, what are the coordinates of *R*?
  - (c) If we now draw a horizontal line from *R* to the point *S* on the line x = a, show that *S* lies on the graph of *h*.
  - (d) By carrying out the construction of the path *PQRS* for several values of *a*, sketch the graph of *h*.



### 2 • SECTION I.2 A CATALOG OF ESSENTIAL FUNCTIONS

33. If *f* is the function whose graph is shown, use the method of Problem 32 to sketch the graph of *f* ∘ *f*. Start by using the construction for *a* = 0, 0.5, 1, 1.5, and 2. Sketch a rough graph for 0 ≤ *x* ≤ 2. Then use the result of Exercise 66 in Section 1.2 to complete the graph.



# 1.2 ANSWERS



















# S Click here for solutions.

















**19.**  $(f \circ g)(x) = f(x^2) = \sqrt{x^2 - 1}, (-\infty, -1] \cup [1, \infty)$   $(g \circ f)(x) = x - 1, [1, \infty)$   $(f \circ f)(x) = \sqrt{\sqrt{x - 1 - 1}}, [2, \infty)$   $(g \circ g)(x) = x^4, (-\infty, \infty)$  **20.**  $(f \circ g)(x) = 1/(x^3 + 2x), \{x \mid x \neq 0\}$   $(g \circ f)(x) = 1/x^3 + 2/x, \{x \mid x \neq 0\}$   $(f \circ f)(x) = x, \{x \mid x \neq 0\}$  $(g \circ g)(x) = x^9 + 6x^7 + 12x^5 + 10x^3 + 4x, (-\infty, \infty)$ 

21. 
$$(f \circ g)(x) = \frac{-x-1}{2}, \{x \mid x \neq -1\}$$
  
 $(g \circ f)(x) = \frac{2-x}{x}, \{x \mid x \neq 0, 1\}$   
 $(f \circ f)(x) = \frac{x-1}{2-x}, \{x \mid x \neq 1, 2\}$   
 $(g \circ g)(x) = -\frac{1}{x}, \{x \mid x \neq 0, -1\}$   
22.  $(f \circ g)(x) = \sqrt{-x}, (-\infty, 0]$   
 $(g \circ f)(x) = \sqrt{1-\sqrt{x^2-1}}, [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$   
 $(f \circ f)(x) = \sqrt{x^2-2}, (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$   
 $(g \circ g)(x) = \sqrt{1-\sqrt{1-x}}, [0, 1]$   
23.  $(f \circ g)(x) = \sqrt[3]{1-\sqrt{x}}, [0, \infty)$   
 $(f \circ f)(x) = \sqrt[9]{x}, (-\infty, \infty)$   
 $(g \circ f)(x) = 1 - \sqrt[6]{x}, [0, \infty)$   
 $(f \circ f)(x) = \sqrt[9]{x}, (-\infty, \infty)$   
 $(g \circ g)(x) = 1 - \sqrt{1-\sqrt{x}}, [0, 1]$   
24.  $(f \circ g)(x) = \frac{3x-4}{3x-2}, \{x \mid x \neq 2, \frac{2}{3}\}$   
 $(g \circ f)(x) = \frac{-x-2}{3x}, \{x \mid x \neq 0, -\frac{1}{2}\}$   
 $(f \circ f)(x) = \frac{5x+4}{4x+5}, \{x \mid x \neq -\frac{1}{2}, -\frac{5}{4}\}$   
 $(g \circ g)(x) = \frac{x}{4-x}, \{x \mid x \neq 2, 4\}$   
25.  $(f \circ g)(x) = 1/\sqrt{x^2-4x}, (-\infty, 0) \cup (4, \infty)$   
 $(g \circ f)(x) = \frac{1}{x} - \frac{4}{\sqrt{x}}, (0, \infty)$   
 $(f \circ f)(x) = x^{1/4}, (0, \infty)$   
 $(g \circ g)(x) = x^4 - 8x^3 + 12x^2 + 16x, (-\infty, \infty)$ 

26. 
$$(f \circ g \circ h)(x) = \sqrt{x - 1} - 1$$
  
27.  $(f \circ g \circ h)(x) = 1/(x^2 + 2)^3$   
28.  $(f \circ g \circ h)(x) = (\sqrt{x} - 5)^4 + 1$   
29.  $(f \circ g \circ h)(x) = \sqrt{\frac{3\sqrt{x}}{\sqrt{x} - 1}}$   
30.  $g(x) = x - 9, f(x) = x^5$   
31.  $g(t) = \pi t, f(t) = \tan t$   
32. (a)  $P(a, g(a)), Q(g(a), g(a))$  (b)  $(g(a), f(g(a)))$   
(d)  $y = \int_{1}^{1} \int_{$ 

### SOLUTIONS

## 🖪 Click here for exercises.

1.2

- (a) The graph of y = x<sup>8</sup> must be the graph labelled g, because g is the graph of a power function of even degree, as shown in Figure 7.
  - (b) The graph of y = log<sub>8</sub> x must be the graph labelled h, because h is a graph similar to the graphs of logarithmic functions shown in Figure 14.
  - (c) The graph of  $y = 2 + \sin 2x$  must be the graph labelled f, because f is the graph of a periodic function.
- (a) The graph of y = x<sup>7</sup> must be the graph labelled G, because G passes through the origin.
  - (b) The graph of y = 7<sup>x</sup> must be the graph labelled F, because F appears to be an exponential function with y-intercept 1, increasing, and horizontal asymptote y = 0.
  - (c) The graph of y = -1/x must be the graph labelled g, because g has a vertical asymptote at x = 0.
  - (d) The graph of  $y = \sqrt[4]{x-2}$  must be the graph labelled f, because f has domain  $[2, \infty)$ .
- **3.** y = -1/x: Start with the graph of y = 1/x and reflect about the *x*-axis.



4.  $y = 2 - \cos x$ : Start with the graph of  $y = \cos x$ , reflect about the x-axis, and then shift 2 units upward.



5.  $y = \tan 2x$ : Start with the graph of  $y = \tan x$  and compress horizontally by a factor of 2.



**6.**  $y = \sqrt[3]{x+2}$ : Start with the graph of  $y = \sqrt[3]{x}$  and shift 2 units to the left.



7.  $y = \cos(x/2)$ : Start with the graph of  $y = \cos x$  and stretch horizontally by a factor of 2.



8.  $y = x^2 + 2x + 3 = (x^2 + 2x + 1) + 2 = (x + 1)^2 + 2$ : Start with the graph of  $y = x^2$ , shift 1 unit left, and then shift 2 units upward.



9.  $y = \frac{1}{x-3}$ : Start with the graph of y = 1/x and shift 3 units to the right.



10.  $y = -2 \sin \pi x$ : Start with the graph of  $y = \sin x$ , compress horizontally by a factor of  $\pi$ , stretch vertically by a factor of 2, and then reflect about the *x*-axis.



11.  $y = \frac{1}{3} \sin \left(x - \frac{\pi}{6}\right)$ : Start with the graph of  $y = \sin x$ , shift  $\frac{\pi}{6}$  units to the right, and then compress vertically by a factor of 3.



12.  $y = 2 + \frac{1}{x+1}$ : Start with the graph of y = 1/x, shift 1 unit left, and then shift 2 units upward.



**13.**  $y = 1 + 2x - x^2 = -x^2 + 2x + 1 = -(x^2 - 2x + 1) + 1 + 1 = -(x - 1)^2 + 2$ : Start with the graph of  $y = x^2$ , shift 1 unit right, reflect about the *x*-axis, and then shift 2 units upward.





14.  $y = \frac{1}{2}\sqrt{x+4} - 3$ : Start with the graph of  $y = \sqrt{x}$ , shift 4 units to the left and compress vertically by a factor of 2, and then shift 3 units downward.



15.  $y = 2 - \sqrt{x+1}$ : Start with the graph of  $y = \sqrt{x}$ , reflect about the *x*-axis, shift 1 unit to the left, and then shift 2 units upward.



16.  $y = (x - 1)^3 + 2$ : Start with the graph of  $y = x^3$ , shift 1 unit to the right, and then shift 2 units upward.



17. y = ||x| - 1|: Start with the graph of y = |x|, shift 1 unit downward, and then reflect the part of the graph from



18.  $y = |\cos x|$ : Start with the graph of  $y = \cos x$  and reflect the parts of the graph that lie below the *x*-axis about the *x*-axis.



$$\begin{aligned} & \text{19. } f\left(x\right) = \sqrt{x-1}, \, D = [1,\infty); \, g\left(x\right) = x^2, \, D = \mathbb{R}. \\ & \left(f \circ g\right)(x) = f\left(g\left(x\right)\right) = f\left(x^2\right) = \sqrt{x^2 - 1}, \\ & D = \{x \in \mathbb{R} \mid g\left(x\right) \in [1,\infty)\} = (-\infty, -1] \cup [1,\infty). \\ & \left(g \circ f\right)(x) = g\left(f\left(x\right)\right) = g\left(\sqrt{x-1}\right) \\ & = \left(\sqrt{x-1}\right)^2 = x - 1, \, D = [1,\infty). \\ & \left(f \circ f\right)(x) = f\left(f\left(x\right)\right) = f\left(\sqrt{x-1}\right) = \sqrt{\sqrt{x-1} - 1}, \\ & D = \left\{x \in [1,\infty) \mid \sqrt{x-1} \ge 1\right\} = [2,\infty). \\ & \left(g \circ g\right)(x) = g\left(g\left(x\right)\right) = g\left(x^2\right) = \left(x^2\right)^2 = x^4, \, D = \mathbb{R}. \end{aligned}$$

**20.** 
$$f(x) = 1/x, D = \{x \mid x \neq 0\}; g(x) = x^3 + 2x, D = \mathbb{R}$$
  
 $(f \circ g)(x) = f(g(x)) = f(x^3 + 2x) = 1/(x^3 + 2x),$   
 $D = \{x \mid x^3 + 2x \neq 0\} = \{x \mid x \neq 0\}.$   
 $(g \circ f)(x) = g(f(x)) = g(1/x) = 1/x^3 + 2/x,$   
 $D = \{x \mid x \neq 0\}.$   
 $(f \circ f)(x) = f(f(x)) = f(1/x) = \frac{1}{1/x} = x,$   
 $D = \{x \mid x \neq 0\}.$   
 $(g \circ g)(x) = g(g(x)) = g(x^3 + 2x)$   
 $= (x^3 + 2x)^3 + 2(x^3 + 2x)$   
 $= x^9 + 6x^7 + 12x^5 + 10x^3 + 4x, D = \mathbb{R}.$ 

**21.** 
$$f(x) = \frac{1}{x-1}, D = \{x \mid x \neq 1\}; g(x) = \frac{x-1}{x+1}, D = \{x \mid x \neq -1\}.$$
  
 $(f \circ g)(x) = f\left(\frac{x-1}{x+1}\right) = \left(\frac{x-1}{x+1} - 1\right)^{-1}$   
 $= \left(\frac{-2}{x+1}\right)^{-1} = \frac{-x-1}{2}, D = \{x \mid x \neq -1\}.$ 

$$(g \circ f)(x) = g\left(\frac{1}{x-1}\right) = \frac{1/(x-1)-1}{1/(x-1)+1} = \frac{2-x}{x},$$
  

$$D = \{x \mid x \neq 0, 1\}.$$
  

$$(f \circ f)(x) = f\left(\frac{1}{x-1}\right) = \frac{1}{1/(x-1)-1} = \frac{x-1}{2-x},$$
  

$$D = \{x \mid x \neq 1, 2\}.$$
  

$$(g \circ g)(x) = g\left(\frac{x-1}{x+1}\right) = \frac{(x-1)/(x+1)-1}{(x-1)/(x+1)+1} = -\frac{1}{x},$$
  

$$D = \{x \mid x \neq 0, -1\}.$$
  
22.  $f(x) = \sqrt{x^2 - 1}, D = (-\infty, -1] \cup [1, \infty);$   

$$g(x) = \sqrt{1-x}, D = (-\infty, 1].$$
  

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{1-x})$$
  

$$= \sqrt{(\sqrt{1-x})^2 - 1} = \sqrt{-x}.$$

To find the domain of  $(f \circ g)(x)$ , we must find the values of x that are in the domain of g such that g(x)is in the domain of f. In symbols, we have  $D = \{ x \in (-\infty, 1] \mid \sqrt{1 - x} \in (-\infty, -1] \cup [1, \infty) \}.$ First, we concentrate on the requirement that  $\sqrt{1-x} \in (-\infty, -1] \cup [1, \infty)$ . Because  $\sqrt{1-x} \ge 0$ ,  $\sqrt{1-x}$  is not in  $(-\infty, -1]$ . If  $\sqrt{1-x}$  is in  $[1, \infty)$ , then we must have  $\sqrt{1-x} \ge 1 \implies 1-x \ge 1 \implies x \le 0$ . Combining the restrictions  $x \leq 0$  and  $x \in (-\infty, 1]$ , we obtain  $D = (-\infty, 0]$ .  $(g \circ f)(x) = g(f(x)) = g(\sqrt{x^2 - 1}) = \sqrt{1 - \sqrt{x^2 - 1}},$  $D = \{ x \in (-\infty, -1] \cup [1, \infty) \mid \sqrt{x^2 - 1} \in (-\infty, 1] \}.$ Now  $\sqrt{x^2 - 1} \le 1 \implies x^2 - 1 \le 1 \implies x^2 \le 2 \implies$  $|x| \leq \sqrt{2} \Rightarrow -\sqrt{2} \leq x \leq \sqrt{2}$ . Combining this restriction with  $x \in (-\infty, -1] \cup [1, \infty)$ , we obtain  $D = \left[-\sqrt{2}, -1\right] \cup \left[1, \sqrt{2}\right].$  $(f \circ f)(x) = f(f(x)) = f(\sqrt{x^2 - 1})$  $=\sqrt{\left(\sqrt{x^2-1}
ight)^2-1}=\sqrt{x^2-2},$  $D = \{ x \in (-\infty, -1] \cup [1, \infty) \mid \sqrt{x^2 - 1} \in (-\infty, -1] \cup [1, \infty) \}.$ Now  $\sqrt{x^2 - 1} \ge 1 \implies x^2 - 1 \ge 1 \implies x^2 \ge 2 \implies$  $|x| \ge \sqrt{2} \implies x \ge \sqrt{2}$  or  $x \le -\sqrt{2}$ . Combining this restriction with  $x \in (-\infty, -1] \cup [1, \infty)$ , we obtain  $D = (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty).$  $(g \circ g)(x) = g(g(x)) = g(\sqrt{1-x}) = \sqrt{1-\sqrt{1-x}},$  $D = \{x \in (-\infty, 1] \mid \sqrt{1 - x} \in (-\infty, 1]\}.$  Now  $\sqrt{1-x} \le 1 \implies 1-x \le 1 \implies x \ge 0$ . Combining this restriction with  $x \in (-\infty, 1]$ , we obtain D = [0, 1].

$$\begin{aligned} \textbf{23. } f(x) &= \sqrt[3]{x}, D = \mathbb{R}; \ g(x) = 1 - \sqrt{x}, D = [0, \infty). \\ (f \circ g)(x) &= f(g(x)) = f(1 - \sqrt{x}) = \sqrt[3]{1 - \sqrt{x}}, \\ D &= [0, \infty). \\ (g \circ g)(x) &= f(f(x)) = f(\sqrt[3]{x}) = x^{1/9}, D = \mathbb{R}. \\ (g \circ g)(x) &= g(g(x)) = g(1 - \sqrt{x}) = 1 - \sqrt{1 - \sqrt{x}}, \\ D &= \{x \ge 0 \mid 1 - \sqrt{x} \ge 0\} = [0, 1]. \end{aligned} \\ \begin{aligned} \textbf{24. } f(x) &= \frac{x + 2}{2x + 1}, D = \{x \mid x \ne -\frac{1}{2}\}; \ g(x) = \frac{x}{x - 2}, \\ D &= \{x \mid x \ne 2\}. \\ (f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x}{x - 2}\right) = \frac{x/(x - 2) + 2}{2x/(x - 2) + 1} \\ &= \frac{3x - 4}{3x - 2}, D = \{x \mid x \ne 2, \frac{2}{3}\}. \\ (g \circ f)(x) &= g(f(x)) \\ &= g\left(\frac{x + 2}{2x + 1}\right) = \frac{(x + 2)/(2x + 1)}{(x + 2)/(2x + 1) - 2} \\ &= \frac{-x - 2}{3x}, D = \{x \mid x \ne 0, -\frac{1}{2}\}. \\ (f \circ f)(x) &= f(f(x)) \\ &= f\left(\frac{x + 2}{2x + 1}\right) = \frac{(x + 2)/(2x + 1) + 2}{2(x + 2)/(2x + 1) + 1} \\ &= \frac{5x + 4}{4x + 5}, D = \{x \mid x \ne -\frac{1}{2}, -\frac{5}{4}\}. \\ (g \circ g)(x) &= g(g(x)) \\ &= g\left(\frac{x}{x - 2}\right) = \frac{x/(x - 2)}{x/(x - 2) - 2} \\ &= \frac{x}{4 - x}, D = \{x \mid x \ne 2, 4\}. \end{aligned}$$

25. 
$$f(x) = 1/\sqrt{x}, D = (0, \infty); g(x) = x^2 - 4x, D = \mathbb{R}.$$
  
 $(f \circ g)(x) = f(g(x)) = f(x^2 - 4x) = 1/\sqrt{x^2 - 4x},$   
 $D = \{x \mid x^2 - 4x > 0\} = (-\infty, 0) \cup (4, \infty).$   
 $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{x} - \frac{4}{\sqrt{x}},$   
 $D = (0, \infty).$   
 $(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{1/\sqrt{x}}} = x^{1/4},$   
 $D = (0, \infty).$   
 $(g \circ g)(x) = g(g(x)) = g(x^2 - 4x)$   
 $= (x^2 - 4x)^2 - 4(x^2 - 4x)$   
 $= x^4 - 8x^3 + 12x^2 + 16x, D = \mathbb{R}.$   
26.  $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x - 1))$   
 $= f(\sqrt{x - 1}) = \sqrt{x - 1} - 1$   
27.  $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2 + 2))$   
 $= f\left((x^2 + 2)^3\right) = 1/(x^2 + 2)^3$ 

**28.** 
$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x}))$$
  
 $= f(\sqrt{x} - 5) = (\sqrt{x} - 5)^4 + 1$   
**29.**  $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt[3]{x}))$   
 $= f\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right) = \sqrt{\frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}}$ 

- **30.** Let g(x) = x 9 and  $f(x) = x^5$ . Then  $(f \circ g)(x) = (x 9)^5 = F(x)$ .
- **31.** Let  $g(t) = \pi t$  and  $f(t) = \tan t$ . Then  $(f \circ g)(t) = \tan \pi t = u(t)$ .
- **32.** (a) P = (a, g(a)) and Q = (g(a), g(a)) because Q has the same y-value as P and it is on the line y = x.
  - (b) The x-value of Q is g(a); this is also the x-value of R. The y-value of R is therefore f(x-value), that is, f(g(a)). Hence, R = (g(a), f(g(a))).
  - (c) The coordinates of S are (a, f(g(a))) or, equivalently, (a, h(a)).



**33.** We need to plot points only for the first quadrant since we can see that f is an odd function, and we then know that  $f \circ f$  is an odd function, and hence, symmetric with respect to the origin.

