1.6 LIMITS INVOLVING INFINITY

A Click here for answers.

S Click here for solutions.

1–3 ▪ Find the limit.

1. $\displaystyle\lim_{x\to 3}\frac{1}{(x-3)^8}$

2. $\displaystyle\lim_{x\to \pi^-}\csc x$

3. $\displaystyle\lim_{x\to 2^+}\frac{x-1}{x^2(x+2)}$

4–11 ▪ Evaluate the limit and justify each step by indicating the appropriate properties of limits.

4. $\displaystyle\lim_{x\to\infty}\frac{1}{x\sqrt{x}}$

5. $\displaystyle\lim_{x\to\infty}\frac{5+2x}{3-x}$

6. $\displaystyle\lim_{x\to\infty}\frac{x+4}{x^2-2x+5}$

7. $\displaystyle\lim_{t\to\infty}\frac{7t^3+4t}{2t^3-t^2+3}$

8. $\displaystyle\lim_{x\to-\infty}\frac{(1-x)(2+x)}{(1+2x)(2-3x)}$

9. $\displaystyle\lim_{x\to\infty}\sqrt{\frac{2x^2-1}{x+8x^2}}$

10. $\displaystyle\lim_{x\to\infty}\frac{1}{3+\sqrt{x}}$

11. $\displaystyle\lim_{x\to\infty}\frac{\sin^2 x}{x^2}$

12–29 ▪ Find the limit.

12. $\displaystyle\lim_{r\to\infty}\frac{r^4-r^2+1}{r^5+r^3-r}$

13. $\displaystyle\lim_{t\to-\infty}\frac{6t^2+5t}{(1-t)(2t-3)}$

14. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{1+4x^2}}{4+x}$

15. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+4x}}{4x+1}$

16. $\displaystyle\lim_{x\to\infty}\frac{1-\sqrt{x}}{1+\sqrt{x}}$

17. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right)$

18. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

19. $\displaystyle\lim_{x\to\infty}\left(\sqrt{1+x}-\sqrt{x}\right)$

20. $\displaystyle\lim_{x\to\infty}\left(\sqrt[3]{1+x}-\sqrt[3]{x}\right)$

21. $\displaystyle\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)$

22. $\displaystyle\lim_{x\to\infty}\left(x+\sqrt{x}\right)$

23. $\displaystyle\lim_{x\to-\infty}(x^3-5x^2)$

24. $\displaystyle\lim_{x\to\infty}\frac{x^7-1}{x^6+1}$

25. $\displaystyle\lim_{x\to\infty}e^{-x^2}$

26. $\displaystyle\lim_{x\to\infty}(x^2-x^4)$

27. $\displaystyle\lim_{x\to\infty}\frac{x^3-1}{x^4+1}$

28. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}+3}{x+3}$

29. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x-1}}$

30. Guess the value of the limit

$$\lim_{x\to\infty}x^2\sin\frac{5}{x^2}$$

by evaluating $f(x)=x^2\sin(5/x^2)$ for $x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then confirm your guess by evaluating this limit exactly.

1.6 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. ∞ **2.** ∞ **3.** $-\infty$ **4.** 0 **5.** -2 **6.** 0 **7.** $\frac{7}{2}$
8. $\frac{1}{6}$ **9.** $\frac{1}{2}$ **10.** 0 **11.** 0 **12.** 0 **13.** -3 **14.** 2
15. $-\frac{1}{4}$ **16.** -1 **17.** $\frac{3}{2}$ **18.** 0 **19.** 0 **20.** 0
21. $-\frac{1}{2}$ **22.** ∞ **23.** $-\infty$ **24.** ∞ **25.** 0 **26.** $-\infty$
27. 0 **28.** 0 **29.** ∞ **30.** 5

1.6 SOLUTIONS

E Click here for exercises.

1. $\lim\limits_{x\to3}\dfrac{1}{(x-3)^8}=\infty$ because $(x-3)^8\to0$ as $x\to3$ and $\dfrac{1}{(x-3)^8}>0$ whenever $x\neq3$.

2. As $x\to\pi^-$, $\sin x\to0^+$, so $\lim\limits_{x\to\pi^-}\csc x=\infty$.

3. $\lim\limits_{x\to-2^+}\dfrac{x-1}{x^2(x+2)}=-\infty$ since the numerator is negative and the denominator approches 0 from the positive side as $x\to-2^+$.

4. $\lim\limits_{x\to\infty}\dfrac{1}{x\sqrt{x}}=\lim\limits_{x\to\infty}\dfrac{1}{x^{3/2}}=0$ by Theorem 5.

5.
$$\lim_{x\to\infty}\frac{5+2x}{3-x}=\lim_{x\to\infty}\frac{\frac{5}{x}+2}{\frac{3}{x}-1}\overset{(5)}{=}\frac{\lim\limits_{x\to\infty}\left[\frac{5}{x}+2\right]}{\lim\limits_{x\to\infty}\left[\frac{3}{x}-1\right]}\overset{(1,2,3)}{=}\frac{5\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}2}{3\lim\limits_{x\to\infty}\frac{1}{x}-\lim\limits_{x\to\infty}1}=\frac{5(0)+2}{3(0)-1}$$
$=-2$ by Limit Law 9 and Theorem 5.

6.
$$\lim_{x\to\infty}\frac{x+4}{x^2-2x+5}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{4}{x^2}}{1-\frac{2}{x}+\frac{5}{x^2}}\overset{(5)}{=}\frac{\lim\limits_{x\to\infty}\left(\frac{1}{x}+\frac{4}{x^2}\right)}{\lim\limits_{x\to\infty}\left(1-\frac{2}{x}+\frac{5}{x^2}\right)}\overset{(1,2,3)}{=}\frac{\lim\limits_{x\to\infty}\frac{1}{x}+4\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}1-2\lim\limits_{x\to\infty}\frac{1}{x}+5\lim\limits_{x\to\infty}\frac{1}{x^2}}$$
$$=\frac{0+4(0)}{1-2(0)+5(0)}=0$$ by Limit Law 9 and Theorem 5.

7.
$$\lim_{t\to\infty}\frac{7t^3+4t}{2t^3-t^2+3}=\lim_{t\to\infty}\frac{7+\frac{4}{t^2}}{2-\frac{1}{t}+\frac{3}{t^3}}\overset{(5,1,2,3)}{=}\frac{\lim\limits_{t\to\infty}7+4\lim\limits_{t\to\infty}\frac{1}{t^2}}{\lim\limits_{t\to\infty}2-\lim\limits_{t\to\infty}\frac{1}{t}+3\lim\limits_{t\to\infty}\frac{1}{t^3}}=\frac{7+4(0)}{2-0+3(0)}=\frac{7}{2}$$
by Limit Law 9 and Theorem 5.

8.
$$\lim_{x\to-\infty}\frac{(1-x)(2+x)}{(1+2x)(2-3x)}=\lim_{x\to-\infty}\frac{\left[\frac{1}{x}-1\right]\left[\frac{2}{x}+1\right]}{\left[\frac{1}{x}+2\right]\left[\frac{2}{x}-3\right]}=\frac{\left[\lim\limits_{x\to-\infty}\frac{1}{x}-1\right]\left[\lim\limits_{x\to-\infty}\frac{2}{x}+1\right]}{\left[\lim\limits_{x\to-\infty}\frac{1}{x}+2\right]\left[\lim\limits_{x\to-\infty}\frac{2}{x}-3\right]}\overset{(5,4,1,2,7)}{=}\frac{(0-1)(0+1)}{(0+2)(0-3)}=\frac{1}{6}$$

9.
$$\lim_{x\to\infty}\left[\frac{2x^2-1}{x+8x^2}\right]^{1/2}\overset{(11)}{=}\left[\lim_{x\to\infty}\frac{2-\frac{1}{x^2}}{\frac{1}{x}+8}\right]^{1/2}\overset{(5,1,2)}{=}\left[\frac{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}8}\right]^{1/2}=\left(\frac{2-0}{0+8}\right)^{1/2}$$
$=\frac{1}{2}$ by Limit Law 9 and Theorem 5.

10.
$$\lim_{x\to\infty}\frac{1}{3+\sqrt{x}}=\lim_{x\to\infty}\frac{1/\sqrt{x}}{(3/\sqrt{x})+1}\overset{(5,1,3)}{=}\frac{\lim\limits_{x\to\infty}(1/\sqrt{x})}{3\lim\limits_{x\to\infty}(1/\sqrt{x})+\lim\limits_{x\to\infty}1}=\frac{0}{3(0)+1}$$
$=0$ by Theorem 5 with $r=\frac{1}{2}$.

Or: Note that $0<\dfrac{1}{3+\sqrt{x}}<\dfrac{1}{\sqrt{x}}$ and use the Squeeze Theorem.

11. Since $0\le\sin^2x\le1$, we have $0\le\dfrac{\sin^2x}{x^2}\le\dfrac{1}{x^2}$ for all $x\neq0$. By Limit Law 9 and Theorem 5 we have $\lim\limits_{x\to\infty}0=0$ and $\lim\limits_{x\to\infty}\dfrac{1}{x^2}=0$. So, by the Squeeze Theorem, $\lim\limits_{x\to\infty}\dfrac{\sin^2x}{x^2}=0$.

12.
$$\lim_{r\to\infty}\frac{r^4-r^2+1}{r^5+r^3-r}=\lim_{r\to\infty}\frac{\frac{1}{r}-\frac{1}{r^3}+\frac{1}{r^5}}{1+\frac{1}{r^2}-\frac{1}{r^4}}=\frac{\lim\limits_{r\to\infty}\frac{1}{r}-\lim\limits_{r\to\infty}\frac{1}{r^3}+\lim\limits_{r\to\infty}\frac{1}{r^5}}{\lim\limits_{r\to\infty}1+\lim\limits_{r\to\infty}\frac{1}{r^2}-\lim\limits_{r\to\infty}\frac{1}{r^4}}=\frac{0-0+0}{1+0-0}=0$$

13.
$$\lim_{t\to-\infty}\frac{6t^2+5t}{(1-t)(2t-3)}=\lim_{t\to-\infty}\frac{6t^2+5t}{-2t^2+5t-3}=\lim_{t\to-\infty}\frac{6+5/t}{-2+5/t-3/t^2}$$
$$=\frac{\lim\limits_{t\to-\infty}6+5\lim\limits_{t\to-\infty}(1/t)}{\lim\limits_{t\to-\infty}(-2)+5\lim\limits_{t\to-\infty}(1/t)-3\lim\limits_{t\to-\infty}(1/t^2)}=\frac{6+5(0)}{-2+5(0)-3(0)}=-3$$

14.
$$\lim_{x\to\infty}\frac{\sqrt{1+4x^2}}{4+x}=\lim_{x\to\infty}\frac{\sqrt{(1/x^2)+4}}{(4/x)+1}=\frac{\sqrt{0+4}}{0+1}=2$$

15. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+4x}}{4x+1}=\lim_{x\to-\infty}\frac{-\sqrt{1+4/x}}{4+1/x}=\frac{-\sqrt{1+0}}{4+0}=-\frac{1}{4}$

Note: In dividing numerator and denominator by x, we used the fact that for $x<0$, $x=-\sqrt{x^2}$.

16. $\displaystyle\lim_{x\to\infty}\frac{1-\sqrt{x}}{1+\sqrt{x}}=\lim_{x\to\infty}\frac{(1/\sqrt{x})-1}{(1/\sqrt{x})+1}=\frac{0-1}{0+1}=-1$

17. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right)$

$$\begin{aligned}&=\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right)\frac{\sqrt{x^2+3x+1}+x}{\sqrt{x^2+3x+1}+x}\\&=\lim_{x\to\infty}\frac{x^2+3x+1-x^2}{\sqrt{x^2+3x+1}+x}\\&=\lim_{x\to\infty}\frac{3x+1}{\sqrt{x^2+3x+1}+x}\\&=\lim_{x\to\infty}\frac{3+1/x}{\sqrt{1+(3/x)+(1/x^2)}+1}\\&=\frac{3+0}{\sqrt{1+3\cdot0+0}+1}=\frac{3}{2}\end{aligned}$$

18. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

$$\begin{aligned}&=\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}\\&=\lim_{x\to\infty}\frac{\left(x^2+1\right)-\left(x^2-1\right)}{\sqrt{x^2+1}+\sqrt{x^2-1}}\\&=\lim_{x\to\infty}\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}\\&=\lim_{x\to\infty}\frac{2/x}{\sqrt{1+(1/x^2)}+\sqrt{1-(1/x^2)}}\\&=\frac{0}{\sqrt{1+0}+\sqrt{1-0}}=0\end{aligned}$$

19. $\displaystyle\lim_{x\to\infty}\left(\sqrt{1+x}-\sqrt{x}\right)$

$$\begin{aligned}&=\lim_{x\to\infty}\left(\sqrt{1+x}-\sqrt{x}\right)\left(\frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}\right)\\&=\lim_{x\to\infty}\frac{(1+x)-x}{\sqrt{1+x}+\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+x}+\sqrt{x}}\\&=\lim_{x\to\infty}\frac{1/\sqrt{x}}{\sqrt{(1/x)+1}+1}=\frac{0}{\sqrt{0+1}+1}=0\end{aligned}$$

20. Using $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$ with $a=\sqrt[3]{1+x}$ and $b=\sqrt[3]{x}$, we have

$\displaystyle\lim_{x\to\infty}\left(\sqrt[3]{1+x}-\sqrt[3]{x}\right)$

$$\begin{aligned}&=\lim_{x\to\infty}\frac{(1+x)-x}{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}\\&=\lim_{x\to\infty}\frac{1}{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}=0\end{aligned}$$

21. $\displaystyle\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)$

$$\begin{aligned}&=\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\left[\frac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x}\right]\\&=\lim_{x\to-\infty}\frac{x+1}{\left(\sqrt{x^2+x+1}-x\right)}\\&=\lim_{x\to-\infty}\frac{1+(1/x)}{-\sqrt{1+(1/x)+(1/x^2)}-1}\\&=\frac{1+0}{-\sqrt{1+0+0}-1}=-\frac{1}{2}\end{aligned}$$

22. $\displaystyle\lim_{x\to\infty}\left(x+\sqrt{x}\right)=\infty$ since $x\to\infty$ and $\sqrt{x}\to\infty$.

23. $\displaystyle\lim_{x\to-\infty}\left(x^3-5x^2\right)=-\infty$ since $x^3\to-\infty$ and $-5x^2\to-\infty$ as $x\to-\infty$.

Or: $\displaystyle\lim_{x\to-\infty}\left(x^3-5x^2\right)=\lim_{x\to-\infty}x^2(x-5)=-\infty$ since $x^2\to\infty$ and $x-5\to-\infty$.

24. $\displaystyle\lim_{x\to\infty}\frac{x^7-1}{x^6+1}=\lim_{x\to\infty}\frac{1-1/x^7}{(1/x)+(1/x^7)}=\infty$ since $\displaystyle 1-\frac{1}{x^7}\to1$ while $\displaystyle\frac{1}{x}+\frac{1}{x^7}\to0^+$ as $x\to\infty$.

Or: Divide numerator and denominator by x^6 instead of x^7.

25. As $x\to\infty$, $x^2\to\infty$ and $-x^2\to-\infty$. Thus, $\displaystyle\lim_{x\to\infty}e^{-x^2}=\lim_{t\to-\infty}e^t=0$.

26. $\displaystyle\lim_{x\to\infty}\left(x^2-x^4\right)=\lim_{x\to\infty}x^2\left(1-x^2\right)=-\infty$ since $x^2\to\infty$ and $1-x^2\to-\infty$.

27. $\displaystyle\lim_{x\to\infty}\frac{x^3-1}{x^4+1}=\lim_{x\to\infty}\frac{(1/x)-(1/x^4)}{1+(1/x^4)}=\frac{0-0}{1+0}=0$

28. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}+3}{x+3}=\lim_{x\to\infty}\frac{(1/\sqrt{x})+(3/x)}{1+3/x}=\frac{0+0}{1+0}=0$

29. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x}-1}=\lim_{x\to\infty}\frac{x/\sqrt{x}}{\sqrt{x}-1/\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{1-1/x}}=\infty$

since $\sqrt{x}\to\infty$ and $\sqrt{1-1/x}\to1$.

Or: Divide numerator and denominator by x instead of $\sqrt{x}$.

30. If $f(x)=x^2\sin\left(5/x^2\right)$, then a calculator gives the following approximate values: $f(1)=-0.95892$, $f(2)=3.79594$, $f(3)=4.74674$, $f(4)=4.91902$, $f(5)=4.96673$, $f(6)=4.98394$, $f(7)=4.99133$, $f(8)=4.99492$, $f(9)=4.99683$, $f(10)=4.99792$, $f(20)=4.99987$, $f(50)=4.999997$, $f(100)=4.9999998$. It appears that $\displaystyle\lim_{x\to\infty}x^2\sin\left(5/x^2\right)=5$.

Proof: Let $t=\dfrac{1}{x^2}$. Then as $x\to\infty$, $t\to0$ and

$$\lim_{x\to\infty}x^2\sin\frac{5}{x^2}=\lim_{t\to0}\left(\frac{1}{t}\sin5t\right)=5\lim_{t\to0}\frac{\sin5t}{5t}=5.$$