2.6 IMPLICIT DIFFERENTIATION

A Click here for answers.

1–5 ▪
(a) Find y' by implicit differentiation.
(b) Solve the equation explicitly for y and differentiate to get y' in terms of x.
(c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $x^2 + 3x + xy = 5$

2. $\frac{x^2}{2} + \frac{y^2}{4} = 1$

3. $2y^2 + xy = x^2 + 3$

4. $\frac{1}{x} + \frac{1}{y} = 3$

5. $x^2 + xy - y^2 = 3$

6–16 ▪ Find dy/dx by implicit differentiation.

6. $y^5 + 3x^2y^2 + 5x^4 = 12$

7. $x^4 + y^4 = 16$

8. $\frac{y}{x - y} = x^2 + 1$

9. $x\sqrt{1 + y} + y\sqrt{1 + 2x} = 2x$

10. $2xy = (x^2 + y^2)^{3/2}$

11. $x^2 = \frac{y^2}{y^2 - 1}$

12. $\sqrt{x + y} + \sqrt{xy} = 6$

S Click here for solutions.

13. $\sqrt{1 + x^2y^2} = 2xy$

14. $x \sin y + \cos 2y = \cos y$

15. $x \cos y + y \cos x = 1$

16. If $x[f(x)]^3 + xf(x) = 6$ and $f(3) = 1$, find $f'(3)$.

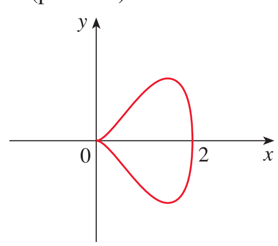
17. If $[g(x)]^2 + 12x = x^2g(x)$ and $g(4) = 12$, find $g'(4)$.

18–21 ▪ Find an equation of the tangent line to the curve at the given point.

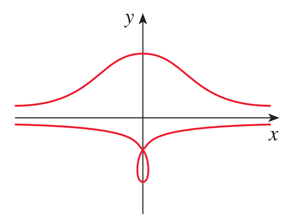
18. $\frac{x^2}{16} - \frac{y^2}{9} = 1, \quad \left(-5, \frac{9}{4}\right)$ (hyperbola)

19. $\frac{x^2}{9} + \frac{y^2}{36} = 1, \quad \left(-1, 4\sqrt{2}\right)$ (ellipse)

20. $y^2 = x^3(2 - x)$
$(1, 1)$
(piriform)



21. $x^2y^2 = (y + 1)^2(4 - y^2)$
$(0, -2)$
(conchoid of Nicomedes)



2.6 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. (a) $y' = -(2x + y + 3)/x$

(b) $y = \dfrac{5}{x} - x - 3,\ y' = -\dfrac{5}{x^2} - 1$

2. (a) $y' = -\dfrac{2x}{y}$

(b) $y = \pm\sqrt{4 - 2x^2},\ y' = \mp\dfrac{2x}{\sqrt{4 - 2x^2}}$

3. (a) $y' = \dfrac{2x - y}{x + 4y}$

(b) $y = \frac{1}{4}\left(-x \pm \sqrt{9x^2 + 24}\right),\ y' = \dfrac{1}{4}\left(-1 \pm \dfrac{9x}{\sqrt{9x^2 + 24}}\right)$

4. (a) $y' = -\dfrac{y^2}{x^2}$

(b) $y = \dfrac{x}{3x - 1},\ y' = -\dfrac{1}{(3x - 1)^2}$

5. (a) $y' = \dfrac{2x + y}{2y - x}$

(b) $y = \frac{1}{2}\left(x \pm \sqrt{5x^2 - 12}\right),\ y' = \dfrac{1}{2}\left(1 \pm \dfrac{5x}{\sqrt{5x^2 - 12}}\right)$

6. $-\dfrac{20x^3 + 6xy^2}{5y^4 + 6x^2y}$ **7.** $-\dfrac{x^3}{y^3}$

8. $\dfrac{y}{x} + 2(x - y)^2$ or $\dfrac{3x^2 + 1 - 2xy}{x^2 + 2}$

9. $\dfrac{2 - \sqrt{1 + y} - y/\sqrt{1 + 2x}}{\sqrt{1 + 2x} + x/\left(2\sqrt{1 + y}\right)}$ **10.** $\dfrac{3x(x^2 + y^2)^{1/2} - 2y}{2x - 3y(x^2 + y^2)^{1/2}}$

11. $-\dfrac{x(y^2 - 1)^2}{y}$ **12.** $-\dfrac{\sqrt{xy} + y\sqrt{x + y}}{\sqrt{xy} + x\sqrt{x + y}}$ **13.** $-\dfrac{y}{x}$

14. $\dfrac{\sin y}{2\sin 2y - x\cos y - \sin y}$ **15.** $\dfrac{y\sin x - \cos y}{\cos x - x\sin y}$

16. $-\frac{1}{6}$ **17.** $\frac{21}{2}$ **18.** $y = -\frac{5}{4}x - 4$

19. $y = \dfrac{1}{\sqrt{2}}(x + 9)$ **20.** $y = x$ **21.** $y = -2$

2.6 SOLUTIONS

E Click here for exercises.

1. (a) $x^2+3x+xy=5 \Rightarrow 2x+3+y+xy'=0 \Rightarrow$
$y'=-\dfrac{2x+y+3}{x}$

(b) $x^2+3x+xy=5 \Rightarrow$
$y=\dfrac{5-x^2-3x}{x}=\dfrac{5}{x}-x-3 \Rightarrow y'=-\dfrac{5}{x^2}-1$

(c) $y'=-\dfrac{2x+y+3}{x}$
$=\dfrac{-2x-3-(-3-x+5/x)}{x}=-1-\dfrac{5}{x^2}$

2. (a) $\dfrac{x^2}{2}+\dfrac{y^2}{4}=1 \Rightarrow x+\dfrac{y}{2}y'=0 \Rightarrow y'=-\dfrac{2x}{y}$

(b) $\dfrac{y^2}{4}=1-\dfrac{x^2}{2} \Rightarrow y^2=4-2x^2$
$\Rightarrow y=\pm\sqrt{4-2x^2} \Rightarrow$
$y'=\pm\dfrac{1}{2\sqrt{4-2x^2}}(-4x)=\mp\dfrac{2x}{\sqrt{4-2x^2}}$

(c) $y'=\dfrac{-2x}{y}=\dfrac{-2x}{\pm\sqrt{4-2x^2}}=\mp\dfrac{2x}{\sqrt{4-2x^2}}$

3. (a) $2y^2+xy=x^2+3 \Rightarrow 4yy'+y+xy'=2x \Rightarrow$
$y'=\dfrac{2x-y}{x+4y}$

(b) Use the quadratic formula:
$2y^2+xy-(x^2+3)=0 \Rightarrow$
$y=\dfrac{-x\pm\sqrt{x^2+8(x^2+3)}}{4}=\dfrac{-x\pm\sqrt{9x^2+24}}{4}$
$\Rightarrow y'=\dfrac{1}{4}\left(-1\pm\dfrac{9x}{\sqrt{9x^2+24}}\right)$

(c) $y'=\dfrac{2x-y}{x+4y}=\dfrac{2x-\frac{1}{4}\left(-x\pm\sqrt{9x^2+24}\right)}{x+\left(-x\pm\sqrt{9x^2+24}\right)}$
$=\dfrac{1}{4}\left(-1\pm\dfrac{9x}{\sqrt{9x^2+24}}\right)$

4. (a) $\dfrac{1}{x}+\dfrac{1}{y}=3 \Rightarrow -\dfrac{1}{x^2}-\dfrac{1}{y^2}y'=0 \Rightarrow y'=-\dfrac{y^2}{x^2}$

(b) $\dfrac{1}{y}=3-\dfrac{1}{x}=\dfrac{3x-1}{x} \Rightarrow y=\dfrac{x}{3x-1} \Rightarrow$
$y'=\dfrac{(3x-1)-(x)(3)}{(3x-1)^2}=-\dfrac{1}{(3x-1)^2}$

(c) $y'=-\dfrac{y^2}{x^2}=-\dfrac{x^2/(3x-1)^2}{x^2}=-\dfrac{1}{(3x-1)^2}$

5. (a) $x^2+xy-y^2=3 \Rightarrow 2x+y+xy'-2yy'=0 \Rightarrow$
$y'=\dfrac{2x+y}{2y-x}$

(b) Use the quadratic formula: $y^2-xs+\left(3-x^2\right)=0 \Rightarrow$
$y=\frac{1}{2}\left[x\pm\sqrt{x^2-4\left(3-x^2\right)}\right]=\frac{1}{2}\left(x\pm\sqrt{5x^2-12}\right)$
$\Rightarrow y'=\dfrac{1}{2}\left(1\pm\dfrac{5x}{\sqrt{5x^2-12}}\right)$

(c) $y'=\dfrac{2x+y}{2y-x}=\dfrac{2x+\frac{1}{2}\left(x\pm\sqrt{5x^2-12}\right)}{x\pm\sqrt{5x^2-12}-x}$
$=\dfrac{1}{2}\left(1\pm\dfrac{5x}{\sqrt{5x^2-12}}\right)$

6. $y^5+3x^2y^2+5x^4=12 \Rightarrow$
$5y^4y'+6xy^2+6x^2yy'+20x^3=0 \Rightarrow$
$y'=-\dfrac{20x^3+6xy^2}{5y^4+6x^2y}$

7. $x^4+y^4=16 \Rightarrow 4x^3+4y^3y'=0 \Rightarrow y'=-\dfrac{x^3}{y^3}$

8. $\dfrac{y}{x-y}=x^2+1 \Rightarrow$
$2x=\dfrac{(x-y)\,y'-y\,(1-y')}{(x-y)^2}=\dfrac{xy'-y}{(x-y)^2} \Rightarrow$
$y'=\dfrac{y}{x}+2\,(x-y)^2.$
Another Method: Write the equation as
$y=(x-y)\left(x^2+1\right)=x^3+x-yx^2-y.$ Then
$y'=\dfrac{3x^2+1-2xy}{x^2+2}.$

9. $x\sqrt{1+y}+y\sqrt{1+2x}=2x \Rightarrow$
$\sqrt{1+y}+x\dfrac{1}{2\sqrt{1+y}}y'+y'\sqrt{1+2x}+y\dfrac{2}{2\sqrt{1+2x}}=2$
$\Rightarrow y'=\dfrac{2-\sqrt{1+y}-\dfrac{y}{\sqrt{1+2x}}}{\sqrt{1+2x}+\dfrac{x}{2\sqrt{1+y}}}$

10. $2xy=\left(x^2+y^2\right)^{3/2} \Rightarrow$
$2y+2xy'=\frac{3}{2}\left(x^2+y^2\right)^{1/2}(2x+2yy') \Rightarrow$
$y'=\dfrac{3x\left(x^2+y^2\right)^{1/2}-2y}{2x-3y\left(x^2+y^2\right)^{1/2}}$

11. $x^2 = \dfrac{y^2}{y^2-1} \quad \Rightarrow$

$2x = \dfrac{(y^2-1)\,2yy' - y^2(2yy')}{(y^2-1)^2} = -\dfrac{2yy'}{(y^2-1)^2} \quad \Rightarrow$

$y' = -\dfrac{x(y^2-1)^2}{y}.$

Another Method: Write the equation as $x^2\left(y^2-1\right) = y^2$.

This gives $y' = \dfrac{x - xy^2}{x^2y - y}$.

12. $\sqrt{x+y} + \sqrt{xy} = 6 \quad \Rightarrow$

$\frac{1}{2}(x+y)^{-1/2}(1+y') + \frac{1}{2}(xy)^{-1/2}(y + xy') = 0 \Rightarrow$

$(x+y)^{-1/2} + (x+y)^{-1/2}y'$

$+ (xy)^{-1/2}y + (xy)^{-1/2}xy' = 0$

$\Rightarrow$

$$y' = -\frac{(x+y)^{-1/2} + (xy)^{-1/2}y}{(x+y)^{-1/2} + (xy)^{-1/2}x} \cdot \frac{(x+y)^{1/2}(xy)^{1/2}}{(x+y)^{1/2}(xy)^{1/2}}$$

$$= -\frac{\sqrt{xy} + y\sqrt{x+y}}{\sqrt{xy} + x\sqrt{x+y}}$$

13. $\sqrt{1+x^2y^2} = 2xy \quad \Rightarrow$

$\frac{1}{2}\left(1+x^2y^2\right)^{-1/2}\left(x^2\cdot 2yy' + y^2\cdot 2x\right) = 2\left(xy' + y\cdot 1\right)$

$\Rightarrow \quad \dfrac{2x^2y}{2\sqrt{1+x^2y^2}}\,y' + \dfrac{2xy^2}{2\sqrt{1+x^2y^2}} = 2xy' + 2y \quad \Rightarrow$

$y'\left(\dfrac{x^2y}{\sqrt{1+x^2y^2}} - 2x\right) = 2y - \dfrac{xy^2}{\sqrt{1+x^2y^2}} \quad \Rightarrow$

$y'\left(\dfrac{x^2y - 2x\sqrt{1+x^2y^2}}{\sqrt{1+x^2y^2}}\right) = \dfrac{2y\sqrt{1+x^2y^2} - xy^2}{\sqrt{1+x^2y^2}} \quad \Rightarrow$

$$y' = \frac{2y\sqrt{1+x^2y^2} - xy^2}{x^2y - 2x\sqrt{1+x^2y^2}} = \frac{y\left(2\sqrt{1+x^2y^2} - xy\right)}{x\left(xy - 2\sqrt{1+x^2y^2}\right)}$$

$$= -\frac{y}{x}$$

Another Method: Since $1 + x^2y^2$ is positive, we can square both sides first and then differentiate implicitly.

14. $x\sin y + \cos 2y = \cos y \quad \Rightarrow$

$\sin y + (x\cos y)\,y' - (2\sin 2y)\,y' = (-\sin y)\,y' \quad \Rightarrow$

$\sin y = (2\sin 2y)\,y' - (x\cos y)\,y' - (\sin y)\,y' \quad \Rightarrow$

$y' = \dfrac{\sin y}{2\sin 2y - x\cos y - \sin y}$

15. $x\cos y + y\cos x = 1 \quad \Rightarrow$

$\cos y + x(-\sin y)\,y' + y'\cos x - y\sin x = 0 \quad \Rightarrow$

$y' = \dfrac{y\sin x - \cos y}{\cos x - x\sin y}$

16. $x\left[f(x)\right]^3 + xf(x) = 6 \quad \Rightarrow$

$[f(x)]^3 + 3x[f(x)]^2 f'(x) + f(x) + xf'(x) = 0 \quad \Rightarrow$

$f'(x) = -\dfrac{[f(x)]^3 + f(x)}{3x[f(x)]^2 + x} \quad \Rightarrow$

$f'(3) = -\dfrac{(1)^3 + 1}{3(3)(1)^2 + 3} = -\dfrac{1}{6}$

17. $[g(x)]^2 + 12x = x^2g(x) \quad \Rightarrow$

$2g(x)\,g'(x) + 12 = 2xg(x) + x^2g'(x)$

$\Leftrightarrow \quad g'(x) = \dfrac{2xg(x) - 12}{2g(x) - x^2} \quad \Rightarrow$

$g'(4) = \dfrac{2(4)(12) - 12}{2(12) - (4)^2} = \dfrac{21}{2}$

18. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1 \quad \Rightarrow \quad \dfrac{x}{8} - \dfrac{2yy'}{9} = 0 \quad \Rightarrow \quad y' = \dfrac{9x}{16y}$.

When $x = -5$ and $y = \frac{9}{4}$ we have $y' = \dfrac{9(-5)}{16(9/4)} = -\dfrac{5}{4}$ so an equation of the tangent is $y - \frac{9}{4} = -\frac{5}{4}(x+5)$ or $y = -\frac{5}{4}x - 4$.

19. $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1 \quad \Rightarrow \quad \dfrac{2x}{9} + \dfrac{yy'}{18} = 0 \quad \Rightarrow \quad y' = -\dfrac{4x}{y}$.

When $x = -1$ and $y = 4\sqrt{2}$ we have $y' = -\dfrac{4(-1)}{4\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ so an equation of the tangent line is $y - 4\sqrt{2} = \frac{1}{\sqrt{2}}(x+1)$ or $y = \frac{1}{\sqrt{2}}(x+9)$.

20. $y^2 = x^3(2-x) = 2x^3 - x^4 \quad \Rightarrow \quad 2yy' = 6x^2 - 4x^3$

$\Rightarrow \quad y' = \dfrac{3x^2 - 2x^3}{y}$. When $x = y = 1$,

$y' = \dfrac{3(1)^2 - 2(1)^3}{1} = 1$, so an equation of the tangent line is $y - 1 = 1(x-1)$ or $y = x$.

21. $x^2y^2 = (y+1)^2\left(4-y^2\right) \quad \Rightarrow$

$2xy^2 + 2x^2yy' = 2(y+1)\,y'\left(4-y^2\right) + (y+1)^2(-2yy')$

$\Rightarrow \quad y' = \dfrac{xy^2}{(y+1)(4-y^2) - y(y+1)^2 - x^2y} = 0$ when $x = 0$. So an equation of the tangent line at $(0, -2)$ is $y + 2 = 0(x-0)$ or $y = -2$.