2.8 LINEAR APPROXIMATIONS AND DIFFERENTIALS

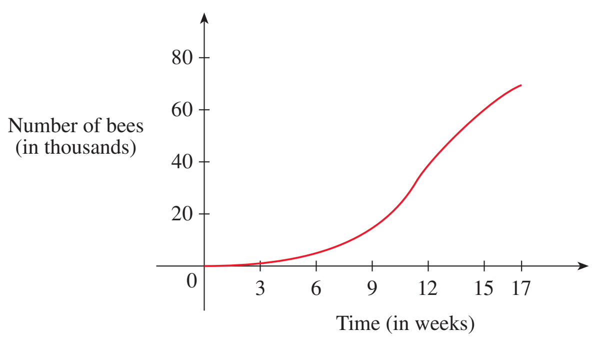
A Click here for answers.

S Click here for solutions.

1. The table lists the amount of U.S. cash per capita in circulation as of June 30 in the given year. Use a linear approximation to estimate the amount of cash per capita in circulation in the year 2000. Is your prediction an underestimate or an overestimate? Why?

Year	1960	1970	1980	1990
Cash per capita	\$177	\$265	\$571	\$1063

2. The figure shows the graph of a population of Cyprian honeybees raised in an apiary.
(a) Use a linear approximation to predict the bee population after 18 weeks and after 20 weeks.
(b) Are your predictions underestimates or overestimates? Why?
(c) Which of your predictions do you think is the more accurate? Why?



3–4 ▪ Find the linearization $L(x)$ of the function at a.

3. $f(x) = 1/\sqrt{2 + x}, \quad a = 0$

4. $f(x) = 1/x \quad a = 4$

5–7 ▪ Verify the given linear approximation at $a = 0$.

5. $\sqrt{1 + x} \approx 1 + \frac{1}{2}x$

6. $\sin x \approx x$

7. $1/\sqrt{4 - x} \approx \frac{1}{2} + \frac{1}{16}x$

8. Verify the linear approximation $\sqrt{1 + x} \approx 1 + \frac{1}{2}x$ at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

9–12 ▪ (a) Find the differential dy and (b) evaluate dy for the given values of x and dx.

9. $y = (x^2 + 5)^3, \quad x = 1,\ dx = 0.05$

10. $y = \sqrt{1 - x}, \quad x = 0,\ dx = 0.02$

11. $y = \cos x, \quad x = \pi/6,\ dx = 0.05$

12. $y = \sin x, \quad x = \pi/6,\ dx = -0.1$

13–17 ▪ Use differentials (or, equivalently, a linear approximation) to estimate the given number.

13. $\sqrt{36.1}$

14. $\sqrt[3]{1.02} + \sqrt[4]{1.02}$

15. $\frac{1}{10.1}$

16. $(1.97)^6$

17. $\sin 59°$

2.8 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. \$1555; underestimate

2. (a) 72,500; 77,500
(b) Overestimates; tangent above graph
(c) 18 weeks; closer to given data

3. $L(x) = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{4}x\right)$ **4.** $L(x) = \frac{1}{2} - \frac{1}{16}x$

8. $-0.69 < x < 1.09$

9. (a) $dy = 6x(x^2 + 5)^2\,dx$ (b) 10.8

10. (a) $dy = -\dfrac{1}{2\sqrt{1-x}}\,dx$ (b) -0.01

11. (a) $dy = -\sin x\,dx$ (b) -0.025

12. (a) $dy = \cos x\,dx$ (b) $-\dfrac{\sqrt{3}}{20}$

13. $\sqrt{36} + \frac{1}{120} \approx 6.0083$ **14.** $2 + \frac{7}{12}(0.02) \approx 2.0117$

15. 0.099 **16.** 58.24 **17.** $\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{360} \approx 0.857$

2.8 SOLUTIONS

E Click here for exercises.

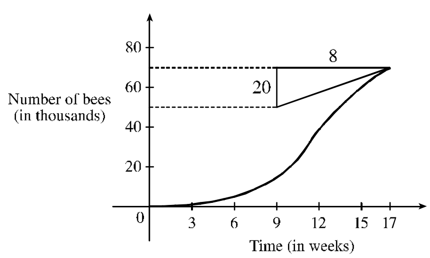
1. If $C(t)$ represents the cash per capita in circulation in year t, then we estimate $C(2000)$ using a linear approximation based on the tangent line to the graph of C at $(1990, C(1990)) = (1990, 1063)$.

$$C'(1990) \approx \frac{C(1980) - C(1990)}{1980 - 1990} = \frac{571 - 1063}{-10} = 49.2\ \frac{\text{dollars}}{\text{year}}$$

$$C(2000) \approx C(1990) + C'(1990)(2000 - 1990) \approx 1063 + 49.2(10) = 1555$$

So our estimate of cash per capita in circulation in the year 2000 is \$1555. For the given data, C' is increasing, so the tangent line approximations are below the curve, indicating that our prediction is an underestimate.

2. (a)



From the figure, $P'(17) \approx \frac{20}{8} = 2.5$ thousand bees/week.

$$P(18) \approx P(17) + P'(17)(18 - 17) \approx 70 + 2.5(1) = 72.5 \text{ or } 72{,}500 \text{ bees}$$

$$P(20) \approx P(17) + P'(17)(20 - 17) \approx 70 + 2.5(3) = 77.5 \text{ or } 77{,}500 \text{ bees}$$

(b) Since the tangent line at $t = 17$ is above the graph, our predictions are overestimates.

(c) $P(18)$ is more accurate than $P(20)$ since it is closer to the given data.

3. $f(x) = 1/\sqrt{2+x} = (2+x)^{-1/2} \Rightarrow$ $f'(x) = -\frac{1}{2}(2+x)^{-3/2}$, so $f(0) = \frac{1}{\sqrt{2}}$ and $f'(0) = -\frac{1}{4\sqrt{2}}$. Thus,

$$L(x) = f(0) + f'(0)(x - 0) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(x - 0) = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{4}x\right)$$

4. $f(x) = 1/x \Rightarrow f'(x) = -1/x^2$. So $f(4) = \frac{1}{4}$ and $f'(4) = -\frac{1}{16}$. Thus,

$$L(x) = f(4) + f'(4)(x - 4) = \frac{1}{4} + \left(-\frac{1}{16}\right)(x - 4) = \frac{1}{2} - \frac{1}{16}x$$

5. $f(x) = \sqrt{1+x} \Rightarrow f'(x) = \frac{1}{2\sqrt{1+x}}$, so $f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{1}{2}x$.

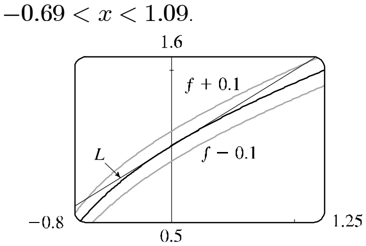
6. $f(x) = \sin x \Rightarrow f'(x) = \cos x$, so $f(0) = 0$ and $f'(0) = 1$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1(x - 0) = x$.

7. $f(x) = \frac{1}{\sqrt{4-x}} \Rightarrow f'(x) = \frac{1}{2(4-x)^{3/2}}$, so $f(0) = \frac{1}{2}$ and $f'(0) = \frac{1}{16}$. Thus, $f(x) \approx \frac{1}{2} + \frac{1}{16}(x - 0) = \frac{1}{2} + \frac{1}{16}x$.

8. $f(x) = \sqrt{1+x} \Rightarrow f'(x) = \frac{1}{2\sqrt{1+x}}$ so $f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus,

$$f(x) \approx f(0) + f'(0)(x - 0) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{1}{2}x$$

We need $\sqrt{1+x} - 0.1 < 1 + \frac{1}{2}x < \sqrt{1+x} + 0.1$. By zooming in or using an intersect feature, we see that this is true when $-0.69 < x < 1.09$.



9. (a) $y = (x^2+5)^3 \Rightarrow$ $dy = 3(x^2+5)^2\, 2x\, dx = 6x(x^2+5)^2\, dx$

(b) When $x = 1$ and $dx = 0.05$, $dy = 6(1)(1^2+5)^2(0.05) = 10.8$.

10. (a) $y = \sqrt{1-x} \Rightarrow$ $dy = \frac{1}{2}(1-x)^{-1/2}(-1)\, dx = -\frac{1}{2\sqrt{1-x}}\, dx$

(b) When $x = 0$ and $dx = 0.02$, $dy = -\frac{1}{2}(0.02) = -0.01$.

11. (a) $y = \cos x \Rightarrow dy = -\sin x\, dx$

(b) When $x = \frac{\pi}{6}$ and $dx = 0.05$, $dy = -\frac{1}{2}(0.05) = -0.025$.

12. (a) $y = \sin x \quad \Rightarrow \quad dy = \cos x \, dx$

(b) When $x = \frac{\pi}{6}$ and $dx = -0.1$, $dy = \frac{\sqrt{3}}{2}(-0.1) = -\frac{\sqrt{3}}{20}$.

13. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad dy = \dfrac{1}{2\sqrt{x}}\, dx$. When $x = 36$ and $dx = 0.1$, $dy = \frac{1}{2\sqrt{36}}(0.1) = \frac{1}{120}$, so

$\sqrt{36.1} = f(36.1) \approx f(36) + dy = \sqrt{36} + \frac{1}{120} \approx 6.0083$.

14. $y = f(x) = \sqrt[3]{x} + \sqrt[4]{x} \quad \Rightarrow$

$dy = \left(\frac{1}{3}x^{-2/3} + \frac{1}{4}x^{-3/4}\right) dx$. If $x = 1$ and $dx = 0.02$, then $dy = \left(\frac{1}{3} + \frac{1}{4}\right)(0.02) = \frac{7}{12}(0.02)$. Thus,

$$\begin{aligned} \sqrt[3]{1.02} + \sqrt[4]{1.02} &= f(1.02) \approx f(1) + dy \\ &= 2 + \tfrac{7}{12}(0.02) \approx 2.0117 \end{aligned}$$

15. $y = f(x) = 1/x \quad \Rightarrow \quad dy = \left(-1/x^2\right) dx$. When $x = 10$ and $dx = 0.1$, $dy = \left(-\frac{1}{100}\right)(0.1) = -0.001$, so

$\frac{1}{10.1} = f(10.1) \approx f(10) + dy = 0.1 - 0.001 = 0.099$.

16. $y = f(x) = x^6 \quad \Rightarrow \quad dy = 6x^5\, dx$. When $x = 2$ and $dx = -0.03$, $dy = 6(2)^5(-0.03) = -5.76$, so

$(1.97)^6 = f(1.97) \approx f(2) + dy = 64 - 5.76 = 58.24$.

17. $y = f(x) = \sin x \quad \Rightarrow \quad dy = \cos x\, dx$. When $x = \frac{\pi}{3}$ and $dx = -\frac{\pi}{180}$, $dy = \cos\frac{\pi}{3}\left(-\frac{\pi}{180}\right) = -\frac{\pi}{360}$, so

$\sin 59^\circ = f\left(\frac{59}{180}\pi\right) \approx f\left(\frac{\pi}{3}\right) + dy = \frac{\sqrt{3}}{2} - \frac{\pi}{360} \approx 0.857$.