6.1 INTEGRATION BY PARTS

A Click here for answers.

S Click here for solutions.

1–15 ▪ Evaluate the integral.

1. $\int xe^{2x}\,dx$

2. $\int x\cos x\,dx$

3. $\int x\sin 4x\,dx$

4. $\int x^2\cos 3x\,dx$

5. $\int x^2\sin ax\,dx$

6. $\int \theta\sin\theta\cos\theta\,d\theta$

7. $\int t^2\ln t\,dt$

8. $\int e^{-\theta}\cos 3\theta\,d\theta$

9. $\int_0^1 te^{-1}\,dt$

10. $\int_1^4 \ln\sqrt{x}\,dx$

11. $\int_0^{\pi/2} x\cos 2x\,dx$

12. $\int_0^1 x^2e^{-x}\,dx$

13. $\int x^3e^{x^2}\,dx$

14. $\int \sin(\ln x)\,dx$

15. $\int x\tan^{-1}x\,dx$

16. First make a substitution and then use integration by parts to evaluate $\int x^5\cos(x^3)\,dx$.

17. Evaluate $\int\sqrt{x}\ln x\,dx$. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

18. Find the area of the region bounded by $y = \sin^{-1}x$, $y = 0$, and $x = 0.5$.

19–20 ▪ Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

19. $y = x^2$, $y = xe^{-x/2}$

20. $y = x^2 - 5$, $y = \ln x$

21. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = \sin x$, $y = 0$, $x = 2\pi$, and $x = 3\pi$ about the y-axis.

22. Find the average value of $f(x) = x\cos 2x$ on the interval $[0, \pi/2]$.

6.1 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$
2. $x\sin x + \cos x + C$
3. $-\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$
4. $\frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$
5. $-\dfrac{x^2}{a}\cos ax + \dfrac{2x}{a^2}\sin ax + \dfrac{2}{a^3}\cos ax + C$
6. $\frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta) + C$
7. $\frac{1}{9}t^3(3\ln t - 1) + C$
8. $\frac{1}{10}e^{-\theta}(3\sin 3\theta - \cos 3\theta) + C$
9. $1 - 2/e$
10. $2\ln 4 - \frac{3}{2}$
11. $-\frac{1}{2}$
12. $2 - 5/e$
13. $\frac{1}{2}e^{x^2}\left(x^2 - 1\right) + C$
14. $\frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$
15. $\frac{1}{2}\left(x^2\tan^{-1}x + \tan^{-1}x - x\right) + C$
16. $\frac{1}{3}x^3\sin\left(x^3\right) + \frac{1}{3}\cos\left(x^3\right) + C$
17. $\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$
18. $\frac{1}{12}\left(\pi + 6\sqrt{3} - 12\right)$
19. 0.080
20. 7.10
21. $10\pi^2$
22. $-\dfrac{1}{\pi}$

6.1 SOLUTIONS

E Click here for exercises.

1. Let $u = x$, $dv = e^{2x}\,dx \Rightarrow du = dx$, $v = \frac{1}{2}e^{2x}$. Then by Equation 2, $\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

2. Let $u = x$, $dv = \cos x\,dx \Rightarrow du = dx$, $v = \sin x$. Then by Equation 2, $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$.

3. Let $u = x$, $dv = \sin 4x\,dx \Rightarrow du = dx$, $v = -\frac{1}{4}\cos 4x$. Then

$$\begin{aligned}\int x\sin 4x\,dx &= -\tfrac{1}{4}x\cos 4x - \int\left(-\tfrac{1}{4}\cos 4x\right)dx\\ &= -\tfrac{1}{4}x\cos 4x + \tfrac{1}{16}\sin 4x + C\end{aligned}$$

4. Let $u = x^2$, $dv = \cos 3x\,dx \Rightarrow du = 2x\,dx$, $v = \frac{1}{3}\sin 3x$. Then $I = \int x^2\cos 3x\,dx = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\int x\sin 3x\,dx$ by Equation 2. Next let $U = x$, $dV = \sin 3x\,dx \Rightarrow dU = dx$, $V = -\frac{1}{3}\cos 3x$ to get

$$\begin{aligned}\int x\sin 3x\,dx &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\int\cos 3x\,dx\\ &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + C_1\end{aligned}$$

Substituting for $\int x\sin 3x\,dx$, we get

$$\begin{aligned}I &= \tfrac{1}{3}x^2\sin 3x - \tfrac{2}{3}\left(-\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + C_1\right)\\ &= \tfrac{1}{3}x^2\sin 3x + \tfrac{2}{9}x\cos 3x - \tfrac{2}{27}\sin 3x + C\end{aligned}$$

where $C = -\frac{2}{3}C_1$.

5. Let $u = x^2$, $dv = \sin ax\,dx \Rightarrow du = 2x\,dx$, $v = -\dfrac{1}{a}\cos ax$. Then

$$\begin{aligned}I &= \int x^2\sin ax\,dx\\ &= -\frac{x^2}{a}\cos ax - \int\left(-\frac{1}{a}\right)\cos ax\,(2x\,dx)\\ &= -\frac{x^2}{a}\cos ax + \frac{2}{a}\int x\cos ax\,dx\end{aligned}$$

by Equation 2. Let $U = x$, $dV = \cos ax\,dx \Rightarrow dU = dx$, $V = \dfrac{1}{a}\sin ax$. Then

$$\begin{aligned}\int x\cos ax\,dx &= \frac{x}{a}\sin ax - \int\frac{1}{a}\sin ax\,dx\\ &= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax + C_1\end{aligned}$$

So

$$\begin{aligned}I &= -\frac{x^2}{a}\cos ax + \frac{2}{a}\left(\frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax + C_1\right)\\ &= -\frac{x^2}{a}\cos ax + \frac{2x}{a^2}\sin ax + \frac{2}{a^3}\cos ax + C\end{aligned}$$

6. $I = \int \theta\sin\theta\cos\theta\,d\theta = \frac{1}{4}\int 2\theta\sin 2\theta\,d\theta$
$= \frac{1}{8}\int t\sin t\,dt$ (Put $t = 2\theta \Rightarrow dt = d\theta/2$.)
Let $u = t$, $dv = \sin t\,dt \Rightarrow du = dt$, $v = -\cos t$. Then

$$\begin{aligned}I &= \tfrac{1}{8}\left(-t\cos t + \int\cos t\,dt\right)\\ &= \tfrac{1}{8}\left(-t\cos t + \sin t\right) + C\\ &= \tfrac{1}{8}\left(\sin 2\theta - 2\theta\cos 2\theta\right) + C\end{aligned}$$

7. Let $u = \ln t$, $dv = t^2\,dt \Rightarrow du = dt/t$, $v = \frac{1}{3}t^3$. Then $\int t^2\ln t\,dt = \frac{1}{3}t^3\ln t - \int\frac{1}{3}t^3\,(1/t)\,dt = \frac{1}{3}t^3\ln t - \frac{1}{9}t^3 + C = \frac{1}{9}t^3\,(3\ln t - 1) + C$.

8. Let $u = \cos 3\theta$, $dv = e^{-\theta}\,d\theta \Rightarrow du = -3\sin 3\theta\,d\theta$, $v = -e^{-\theta}$. Then $I = \int e^{-\theta}\cos 3\theta\,d\theta = -e^{-\theta}\cos 3\theta - 3\int e^{-\theta}\sin 3\theta\,d\theta$. Integrate by parts again: $I = -e^{-\theta}\cos 3\theta + 3e^{-\theta}\sin 3\theta - \int e^{-\theta}9\cos 3\theta\,d\theta$, so $10\int e^{-\theta}\cos 3\theta\,d\theta = e^{-\theta}\,(3\sin 3\theta - \cos 3\theta) + C_1$ and $I = \frac{1}{10}e^{-\theta}\,(3\sin 3\theta - \cos 3\theta) + C$, where $C = C_1/10$.

9. Let $u = t$, $dv = e^{-t}\,dt \Rightarrow du = dt$, $v = -e^{-t}$. By Formula 6,

$$\begin{aligned}\int_0^1 te^{-t}\,dt &= \left[-te^{-t}\right]_0^1 + \int_0^1 e^{-t}\,dt\\ &= -1/e + \left[-e^{-t}\right]_0^1 = -1/e - 1/e + 1\\ &= 1 - 2/e\end{aligned}$$

10. $I = \int_1^4 \ln\sqrt{x}\,dx = \frac{1}{2}\int_1^4 \ln x\,dx = \frac{1}{2}\left[x\ln x - x\right]_1^4$ as in Example 2. So $I = \frac{1}{2}\left[(4\ln 4 - 4) - (0 - 1)\right] = 2\ln 4 - \frac{3}{2}$.

11. Let $u = x$, $dv = \cos 2x\,dx \Rightarrow du = dx$, $v = \frac{1}{2}\sin 2x\,dx$. Then

$$\begin{aligned}\int_0^{\pi/2} x\cos 2x\,dx &= \left[\tfrac{1}{2}x\sin 2x\right]_0^{\pi/2} - \tfrac{1}{2}\int_0^{\pi/2}\sin 2x\,dx\\ &= 0 + \left[\tfrac{1}{4}\cos 2x\right]_0^{\pi/2}\\ &= \tfrac{1}{4}(-1 - 1) = -\tfrac{1}{2}\end{aligned}$$

12. Let $u = x^2$, $dv = e^{-x}\,dx \Rightarrow du = 2x\,dx$, $v = -e^{-x}$. Then

$$\begin{aligned}I &= \int_0^1 x^2e^{-x}\,dx = \left[-x^2e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx\\ &= -1/e + \int_0^1 2xe^{-x}\,dx\end{aligned}$$

Now use parts again with $u = 2x$, $dv = e^{-x}$. Then

$$\begin{aligned}I &= -1/e - \left[2xe^{-x}\right]_0^1 + \int_0^1 2e^{-x}\,dx\\ &= -1/e - 2/e - \left[2e^{-x}\right]_0^1 = -3/e - 2/e + 2\\ &= 2 - 5/e\end{aligned}$$

13. Substitute $t = x^2 \quad \Rightarrow \quad dt = 2x\,dx$. Then use parts with $u = t$, $dv = e^t\,dt \quad \Rightarrow \quad du = dt$, $v = e^t$. Thus,

$$\begin{aligned} \int x^3 e^{x^2}\,dx &= \tfrac{1}{2}\int te^t\,dt = \tfrac{1}{2}te^t - \tfrac{1}{2}\int e^t\,dt \\ &= \tfrac{1}{2}te^t - \tfrac{1}{2}e^t + C = \tfrac{1}{2}e^{x^2}\left(x^2 - 1\right) + C \end{aligned}$$

14. Let $w = \ln x$, so that $x = e^w$ and $dx = e^w\,dw$. Then

$$\begin{aligned} \int \sin(\ln x)\,dx &= \int e^w \sin w\,dw \\ &= \tfrac{1}{2}e^w(\sin w - \cos w) + C \text{ (by Example 4)} \\ &= \tfrac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C \end{aligned}$$

15. Let $u = \tan^{-1} x$, $dv = x\,dx \quad \Rightarrow \quad du = dx/\left(1 + x^2\right)$, $v = \frac{1}{2}x^2$.

Then $\int x \tan^{-1} x\,dx = \frac{1}{2}x^2 \tan^{-1} x - \dfrac{1}{2}\displaystyle\int \frac{x^2}{1 + x^2}\,dx$.

But

$$\begin{aligned} \int \frac{x^2}{1 + x^2}\,dx &= \int \frac{\left(1 + x^2\right) - 1}{1 + x^2}\,dx \\ &= \int 1\,dx - \int \frac{1}{1 + x^2}\,dx \\ &= x - \tan^{-1} x + C_1 \end{aligned}$$

so

$$\begin{aligned} \int x \tan^{-1} x\,dx &= \tfrac{1}{2}x^2 \tan^{-1} x - \tfrac{1}{2}\left(x - \tan^{-1} x + C_1\right) \\ &= \tfrac{1}{2}\left(x^2 \tan^{-1} x + \tan^{-1} x - x\right) + C \end{aligned}$$

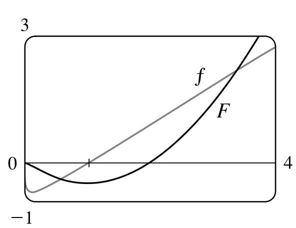
16. Substitute $t = x^3 \quad \Rightarrow \quad dt = 3x^2\,dx$. Then use parts with $u = t$, $dv = \cos t\,dt$. Thus

$$\begin{aligned} \int x^5 \cos\left(x^3\right)dx &= \tfrac{1}{3}\int x^3 \cos\left(x^3\right) \cdot 3x^2\,dx = \tfrac{1}{3}\int t\cos t\,dt \\ &= \tfrac{1}{3}t\sin t - \tfrac{1}{3}\int \sin t\,dt \\ &= \tfrac{1}{3}t\sin t + \tfrac{1}{3}\cos t + C \\ &= \tfrac{1}{3}x^3 \sin\left(x^3\right) + \tfrac{1}{3}\cos\left(x^3\right) + C \end{aligned}$$

17. Let $u = \ln x$, $dv = \sqrt{x}\,dx \quad \Rightarrow \quad du = dx/x$, $v = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2}$. Thus

$$\begin{aligned} \int \sqrt{x}\ln x\,dx &= \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{3/2}(1/x)\,dx \\ &= \tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2} + C \end{aligned}$$

We see from the graph that this is reasonable, since the antiderivative is increasing where the original function is positive.

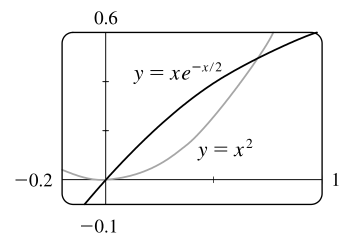


18. Let $u = \sin^{-1} x$, $dv = dx \quad \Rightarrow \quad du = \dfrac{dx}{\sqrt{1 - x^2}}$, $v = x$.

Then

$$\begin{aligned} \text{area} &= \int_0^{1/2} \sin^{-1} x\,dx \\ &= \left[x\sin^{-1} x\right]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1 - x^2}}\,dx \\ &= \tfrac{1}{2}\left(\tfrac{\pi}{6}\right) + \left[\sqrt{1 - x^2}\right]_0^{1/2} \\ &= \tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2} - 1 = \tfrac{1}{12}\left(\pi + 6\sqrt{3} - 12\right) \end{aligned}$$

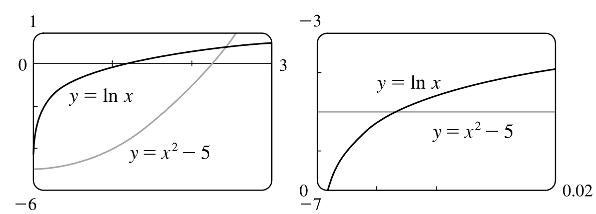
19.



From the graph, we see that the curves intersect at approximately $x = 0$ and $x = 0.70$, with $xe^{-x/2} > x^2$ on $(0, 0.70)$. So the area bounded by the curves is approximately $A = \int_0^{0.70}\left(xe^{-x/2} - x^2\right)dx$. We separate this into two integrals, and evaluate the first one by parts with $u = x$, $dv = e^{-x/2}\,dx \quad \Rightarrow \quad du = dx$, $v = -2e^{-x/2}$:

$$\begin{aligned} A &= \left[-2xe^{-x/2}\right]_0^{0.70} - \int_0^{0.70}\left(-2e^{-x/2}\right)dx - \left[\tfrac{1}{3}x^3\right]_0^{0.70} \\ &= \left[-2(0.70)e^{-0.35} - 0\right] - \left[4e^{-x/2}\right]_0^{0.70} - \tfrac{1}{3}\left[0.70^3 - 0\right] \\ &\approx 0.080 \end{aligned}$$

20.



From the graphs, we see that the curves intersect at approximately $x = 0.0067$ and $x = 2.43$, with $\ln x > x^2 - 5$ on $(0.0067, 2.43)$. So the area bounded by the curves is about

$$\begin{aligned} A &= \int_{0.0067}^{2.43}\left[\ln x - \left(x^2 - 5\right)\right]dx \\ &= \int_{0.0067}^{2.43}\left(\ln x - x^2 + 5\right)dx \\ &= \left[(x\ln x - x) - \tfrac{1}{3}x^3 + 5x\right]_{0.0067}^{2.43} \text{ (see Example 2)} \\ &\approx 7.10 \end{aligned}$$

21. Volume $= \int_{2\pi}^{3\pi} 2\pi x \sin x\, dx$. Let $u = x$, $dv = \sin x\, dx \quad \Rightarrow$
$du = dx$, $v = -\cos x \quad \Rightarrow$

$$\begin{aligned} V &= 2\pi \left[-x\cos x + \sin x\right]_{2\pi}^{3\pi} \\ &= 2\pi \left[(3\pi + 0) - (-2\pi + 0)\right] \\ &= 2\pi\,(5\pi) = 10\pi^2 \end{aligned}$$

22. Let $u = x$, $dv = \cos 2x\, dx \quad \Rightarrow \quad du = dx$,
$v = \frac{1}{2}\sin 2x\, dx$. Then

$$\begin{aligned} \int_0^{\pi/2} x\cos 2x\, dx &= \left[\tfrac{1}{2}x\sin 2x\right]_0^{\pi/2} - \tfrac{1}{2}\int_0^{\pi/2} \sin 2x\, dx \\ &= 0 + \left[\tfrac{1}{4}\cos 2x\right]_0^{\pi/2} \\ &= \tfrac{1}{4}(-1-1) = -\tfrac{1}{2} \end{aligned}$$

Hence, the average value of f is $\dfrac{-1/2}{\pi/2 - 0} = -\dfrac{1}{\pi}$.