6.2 TRIGONOMETRIC INTEGRALS AND SUBSTITUTIONS

A Click here for answers.

S Click here for solutions.

1–56 ▪ Evaluate the integral.

1. $\int \sin^3 x\,dx$

2. $\int \sin^3 x \cos^4 x\,dx$

3. $\int \sin^4 x \cos^3 x\,dx$

4. $\int \cos^5 x \sin^5 x\,dx$

5. $\int_0^{\pi/2} \sin^2 3x\,dx$

6. $\int_0^{\pi/2} \cos^2 x\,dx$

7. $\int \cos^4 t\,dt$

8. $\int \sin^6 \pi x\,dx$

9. $\int (1 - \sin 2x)^2\,dx$

10. $\int \sin\left(\theta + \dfrac{\pi}{6}\right) \cos\theta\,d\theta$

11. $\int x \sin^3(x^2)\,dx$

12. $\int \cos^6 x\,dx$

13. $\int \sin^5 2x \cos^4 2x\,dx$

14. $\int \sin^5 x\,dx$

15. $\int \sin^4 x \cos^4 x\,dx$

16. $\int \dfrac{\cos^3 x}{\sqrt{\sin x}}\,dx$

17. $\int \dfrac{\cos^2(\sqrt{x})}{\sqrt{x}}\,dx$

18. $\int \dfrac{dx}{1 - \sin x}$

19. $\int \tan x \sec^6 x\,dx$

20. $\int \tan^3 x \sec^6 x\,dx$

21. $\int \sec^4 x\,dx$

22. $\int_0^{\pi/4} \sec^6 x\,dx$

23. $\int_0^{\pi/4} \tan^4 t \sec^2 t\,dt$

24. $\int_0^{\pi/4} \tan^2 x \sec^4 x\,dx$

25. $\int \tan^3 x \sec^3 x\,dx$

26. $\int_0^{\pi/3} \tan^5 x \sec x\,dx$

27. $\int \dfrac{\sec^2 x}{\cot x}\,dx$

28. $\int \cot^2 w \csc^4 w\,dw$

29. $\int \cot^3 x \csc^4 x\,dx$

30. $\int e^x \cos^7(e^x)\,dx$

31. $\int \dfrac{\cos^2 x}{\sin x}\,dx$

32. $\int \dfrac{dx}{\sin^4 x}$

33. $\int \cos 3x \cos 4x\,dx$

34. $\int \sin 3x \sin 6x\,dx$

35. $\int \sin x \cos 5x\,dx$

36. $\int \cos x \cos 2x \cos 3x\,dx$

37. $\int_{1/2}^{\sqrt{3}/2} \dfrac{1}{x^2\sqrt{1 - x^2}}\,dx$

38. $\int_0^2 x^3\sqrt{4 - x^2}\,dx$

39. $\int \dfrac{x}{\sqrt{1 - x^2}}\,dx$

40. $\int x\sqrt{4 - x^2}\,dx$

41. $\int_0^2 \dfrac{x^3}{\sqrt{x^2 + 4}}\,dx$

42. $\int_0^3 \dfrac{dx}{\sqrt{9 + x^2}}$

43. $\int \dfrac{dx}{x^3\sqrt{x^2 - 16}}$

44. $\int \dfrac{x}{(x^2 + 4)^{5/2}}\,dx$

45. $\int \dfrac{dx}{x\sqrt{x^2 + 3}}$

46. $\int x\sqrt{25 + x^2}\,dx$

47. $\int \dfrac{\sqrt{9x^2 - 4}}{x}\,dx$

48. $\int_0^3 x\sqrt{9 - x^2}\,dx$

49. $\int 5x\sqrt{1 + x^2}\,dx$

50. $\int \dfrac{dx}{(4x^2 - 25)^{3/2}}$

51. $\int \dfrac{dx}{\sqrt{x^2 + 4x + 8}}$

52. $\int \sqrt{2x - x^2}\,dx$

53. $\int e^t\sqrt{9 - e^{2t}}\,dt$

54. $\int \sqrt{e^{2t} - 9}\,dt$

55. $\int \dfrac{dx}{x^4\sqrt{x^2 - 2}}$

56. $\int \dfrac{dx}{(1 + x^2)^2}$

57. Find the average value of $f(x) = (4 - x^2)^{3/2}$ on the interval $[0, 2]$.

6.2 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $-\cos x+\frac{1}{3}\cos^3 x+C$

2. $\frac{1}{7}\cos^7 x-\frac{1}{5}\cos^5 x+C$

3. $\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+C$

4. $\frac{1}{10}\sin^{10} x-\frac{1}{4}\sin^8 x+\frac{1}{6}\sin^6 x+C$

5. $\frac{\pi}{4}$ **6.** $\frac{\pi}{4}$

7. $\frac{3}{8}t+\frac{1}{4}\sin 2t+\frac{1}{32}\sin 4t+C$

8. $\frac{5}{16}x-\frac{1}{4\pi}\sin 2\pi x+\frac{3}{64\pi}\sin 4\pi x+\frac{1}{48\pi}\sin^3 2\pi x+C$

9. $\frac{3}{2}x+\cos 2x-\frac{1}{8}\sin 4x+C$

10. $-\frac{\sqrt{3}}{8}\cos 2\theta+\frac{1}{4}\theta+\frac{1}{8}\sin 2\theta+C$

11. $-\frac{1}{2}\cos\left(x^2\right)+\frac{1}{6}\cos^3\left(x^2\right)+C$

12. $\frac{1}{8}\left(\frac{5}{2}x+2\sin 2x+\frac{3}{8}\sin 4x-\frac{1}{6}\sin^3 2x\right)+C$

13. $-\frac{1}{2}\left(\frac{1}{9}\cos^9 2x-\frac{2}{7}\cos^7 2x+\frac{1}{5}\cos^5 2x\right)+C$

14. $-\frac{1}{5}\cos^5 x+\frac{2}{3}\cos^3 x-\cos x+C$

15. $\frac{3}{128}x-\frac{1}{128}\sin 4x+\frac{1}{1024}\sin 8x+C$

16. $2\left(1-\frac{1}{5}\sin^2 x\right)\sqrt{\sin x}+C$

17. $\sqrt{x}+\frac{1}{2}\sin\left(2\sqrt{x}\right)+C$

18. $\tan x+\sec x+C$

19. $\frac{1}{6}\sec^6 x+C$ or $\frac{1}{2}\tan^2 x+\frac{1}{2}\tan^4 x+\frac{1}{2}\tan^6 x+C$

20. $\frac{1}{8}\sec^8 x-\frac{1}{6}\sec^6 x+C$ or
$\frac{1}{8}\tan^2 x+\frac{1}{3}\tan^6 x+\frac{1}{4}\tan^4 x+C$

21. $\frac{1}{3}\tan^3 x+\tan x+C$

22. $\frac{28}{15}$ **23.** $\frac{1}{5}$ **24.** $\frac{8}{15}$

25. $\frac{1}{5}\sec^5 x-\frac{1}{3}\sec^3 x+C$

26. $\frac{38}{15}$

27. $\frac{1}{2}\tan^2 x+C$

28. $-\frac{1}{3}\cot^3 w-\frac{1}{5}\cot^5 w+C$

29. $-\frac{1}{6}\cot^6 x-\frac{1}{4}\cot^4 x+C$

30. $\sin e^x-\sin^3 e^x+\frac{3}{5}\sin^5 e^x-\frac{1}{7}\sin^7 e^x+C$

31. $\ln|\csc x-\cot x|+\cos x+C$

32. $-\frac{1}{3}\cot^3 x-\cot x+C$

33. $\frac{1}{2}\sin x+\frac{1}{14}\sin 7x+C$

34. $\frac{1}{6}\sin 3x-\frac{1}{18}\sin 9x+C$

35. $\frac{1}{8}\cos 4x-\frac{1}{12}\cos 6x+C$

36. $\frac{1}{4}x+\frac{1}{8}\sin 2x+\frac{1}{16}\sin 4x+\frac{1}{24}\sin 6x+C$

37. $\frac{2}{\sqrt{3}}$

38. $\frac{64}{15}$

39. $-\sqrt{1-x^2}+C$

40. $-\frac{1}{3}\left(4-x^2\right)^{3/2}+C$

41. $\frac{8}{3}\left(2-\sqrt{2}\right)$

42. $\ln\left(\sqrt{2}+1\right)$

43. $\frac{1}{128}\left(\sec^{-1}\frac{x}{4}+\frac{4\sqrt{x^2-16}}{x^2}\right)+C$

44. $-\frac{1}{3}\left(x^2+4\right)^{-3/2}+C$

45. $\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+3}-\sqrt{3}}{x}\right|+C$

46. $\frac{1}{3}\left(x^2+25\right)^{3/2}+C$

47. $\sqrt{9x^2-4}-2\sec^{-1}\left(\frac{3x}{2}\right)+C$

48. 9

49. $\frac{5}{3}\left(1+x^2\right)^{3/2}+C$

50. $-\dfrac{x}{25\sqrt{4x^2-25}}+C$

51. $\ln\left(\sqrt{x^2+4x+8}+x+2\right)+C$

52. $\frac{1}{2}\left[\sin^{-1}(x-1)+(x-1)\sqrt{2x-x^2}\right]+C$

53. $\frac{9}{2}\sin^{-1}\left(\frac{1}{3}e^t\right)+\frac{1}{2}e^t\sqrt{9-e^{2t}}+C$

54. $\sqrt{e^{2t}-9}-3\sec^{-1}\left(\frac{1}{3}e^t\right)+C$

55. $\dfrac{1}{4}\left[\dfrac{\sqrt{x^2-2}}{x}-\dfrac{\left(x^2-2\right)^{3/2}}{3x^3}\right]+C$

56. $\frac{1}{2}\left(\tan^{-1}x+\dfrac{x}{x^2+1}\right)+C$

57. $\frac{3\pi}{2}$

6.2 SOLUTIONS

E Click here for exercises.

1. $\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx$
$= \int \sin x\,dx + \int \cos^2 x\,(-\sin x)\,dx$
$= -\cos x + \frac{1}{3}\cos^3 x + C$
by the substitution $u = \cos x$ in the second integral.

2. Let $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx$. Then
$\int \sin^3 x \cos^4 x\,dx = \int \cos^4 x\,(1-\cos^2 x)\sin x\,dx$
$= \int u^4 (1-u^2)(-du) = \int (u^6 - u^4)\,du$
$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$

3. Let $u = \sin x \quad \Rightarrow \quad du = \cos x\,dx$. Then
$\int \sin^4 x \cos^3 x\,dx = \int \sin^4 x\,(1-\sin^2 x)\cos x\,dx$
$= \int (u^4 - u^6)\,du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$
$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

4. Let $u = \sin x \quad \Rightarrow \quad du = \cos x\,dx$. Then
$\int \cos^5 x \sin^5 x\,dx = \int u^5 (1-u^2)^2\,du$
$= \int u^5 (1 - 2u^2 + u^4)\,du = \int (u^5 - 2u^7 + u^9)\,du$
$= \frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6 + C$
$= \frac{1}{10}\sin^{10} x - \frac{1}{4}\sin^8 x + \frac{1}{6}\sin^6 x + C$
Or: Let $v = \cos x$, $dv = -\sin x\,dx$. Then
$\int \cos^5 x \sin^5 x\,dx = \int v^5 (1-v^2)^2 (-dv)$
$= \int (-v^5 + 2v^7 - v^9)\,dv = -\frac{1}{10}v^{10} + \frac{1}{4}v^8 - \frac{1}{6}v^6 + C$
$= -\frac{1}{10}\cos^{10} x + \frac{1}{4}\cos^8 x - \frac{1}{6}\cos^6 x + C$

5. $\int_0^{\pi/2} \sin^2 3x\,dx = \int_0^{\pi/2} \frac{1}{2}(1 - \cos 6x)\,dx$
$= \left[\frac{1}{2}x - \frac{1}{12}\sin 6x\right]_0^{\pi/2} = \frac{\pi}{4}$

6. $\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2x)\,dx$
$= \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{4}$

7. $\int \cos^4 t\,dt = \int \left[\frac{1}{2}(1 + \cos 2t)\right]^2 dt$
$= \frac{1}{4}\int (1 + 2\cos 2t + \cos^2 2t)\,dt$
$= \frac{1}{4}t + \frac{1}{4}\sin 2t + \frac{1}{4}\int \frac{1}{2}(1 + \cos 4t)\,dt$
$= \frac{1}{4}\left[t + \sin 2t + \frac{1}{2}t + \frac{1}{8}\sin 4t\right] + C$
$= \frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C$

8. $\int \sin^6 \pi x\,dx = \int (\sin^2 \pi x)^3\,dx$
$= \int \left[\frac{1}{2}(1 - \cos 2\pi x)\right]^3 dx$
$= \frac{1}{8}\int (1 - 3\cos 2\pi x + 3\cos^2 2\pi x - \cos^3 2\pi x)\,dx$
$= \frac{1}{8}\int \left[1 - 3\cos 2\pi x + \frac{3}{2}(1 + \cos 4\pi x)\right.$
$\left. - (1 - \sin^2 2\pi x)\cos 2\pi x\right] dx$
$= \frac{1}{8}\int \left(\frac{5}{2} - 4\cos 2\pi x + \frac{3}{2}\cos 4\pi x + \sin^2 2\pi x \cos 2\pi x\right) dx$
$= \frac{1}{8}\left[\frac{5}{2}x - \frac{4}{2\pi}\sin 2\pi x + \frac{3}{8\pi}\sin 4\pi x + \frac{1}{3\cdot 2\pi}\sin^3 2\pi x\right] + C$
$= \frac{5}{16}x - \frac{1}{4\pi}\sin 2\pi x + \frac{3}{64\pi}\sin 4\pi x + \frac{1}{48\pi}\sin^3 2\pi x + C$

9. $\int (1 - \sin 2x)^2\,dx = \int (1 - 2\sin 2x + \sin^2 2x)\,dx$
$= \int \left[1 - 2\sin 2x + \frac{1}{2}(1 - \cos 4x)\right] dx$
$= \int \left[\frac{3}{2} - 2\sin 2x - \frac{1}{2}\cos 4x\right] dx$
$= \frac{3}{2}x + \cos 2x - \frac{1}{8}\sin 4x + C$

10. $\int \sin\left(\theta + \frac{\pi}{6}\right)\cos\theta\,d\theta$
$= \int \left(\sin\theta \cdot \frac{\sqrt{3}}{2} + \cos\theta \cdot \frac{1}{2}\right)\cos\theta\,d\theta$
$= \frac{\sqrt{3}}{4}\int \sin 2\theta\,d\theta + \frac{1}{4}\int (1 + \cos 2\theta)\,d\theta$
$= -\frac{\sqrt{3}}{8}\cos 2\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C$

11. Let $u = x^2 \quad \Rightarrow \quad du = 2x\,dx$. Then
$\int x\sin^3(x^2)\,dx = \int \sin^3 u \cdot \frac{1}{2}du$
$= \frac{1}{2}\left(-\cos u + \frac{1}{3}\cos^3 u\right) + C$
$= -\frac{1}{2}\cos(x^2) + \frac{1}{6}\cos^3(x^2) + C$

12. $\int \cos^6 x\,dx = \int \left[\frac{1}{2}(1 + \cos 2x)\right]^3 dx$
$= \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\,dx$
$= \frac{1}{8}\int \left[1 + 3\cos 2x +\right.$
$\left. + \frac{3}{2}(1 + \cos 4x) + (1 - \sin^2 2x)\cos x\right] dx$
$= \frac{1}{8}\int \left(\frac{5}{2} + 4\cos 2x + \frac{3}{2}\cos 4x - \sin^2 2x\cos 2x\right) dx$
$= \frac{1}{8}\left(\frac{5}{2}x + 2\sin 2x + \frac{3}{8}\sin 4x - \frac{1}{6}\sin^3 2x\right) + C$

13. Let $u = \cos 2x \quad \Rightarrow \quad du = -2\sin 2x\,dx$. Then
$\int \sin^5 2x \cos^4 2x\,dx = \int \cos^4 2x\,(1 - \cos^2 2x)^2 \sin 2x\,dx$
$= \int u^4 (1 - u^2)^2 \left(-\frac{1}{2}du\right) = -\frac{1}{2}\int (u^4 - 2u^6 + u^8)\,du$
$= -\frac{1}{2}\left(\frac{1}{9}u^9 - \frac{2}{7}u^7 + \frac{1}{5}u^5\right) + C$
$= -\frac{1}{2}\left(\frac{1}{9}\cos^9 2x - \frac{2}{7}\cos^7 2x + \frac{1}{5}\cos^5 2x\right) + C$

14. Let $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx$. Then
$\int \sin^5 x\,dx = \int (1 - \cos^2 x)^2 \sin x\,dx$
$= \int (1 - u^2)^2 (-du) = \int (-1 + 2u^2 - u^4)\,du$
$= -\frac{1}{5}u^5 + \frac{2}{3}u^3 - u + C$
$= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

15. $\int \sin^4 x \cos^4 x\,dx = \int \left(\frac{1}{2}\sin 2x\right)^4 dx$
$= \frac{1}{16}\int \sin^4 2x\,dx = \frac{1}{16}\int \left[\frac{1}{2}(1 - \cos 4x)\right]^2 dx$
$= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x)\,dx$
$= \frac{1}{64}\left(x - \frac{1}{2}\sin 4x\right) + \frac{1}{128}\int (1 + \cos 8x)\,dx$
$= \frac{1}{64}\left(x - \frac{1}{2}\sin 4x\right) + \frac{1}{128}\left(x + \frac{1}{8}\sin 8x\right) + C$
$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$

16. Let $u = \sin x \quad\Rightarrow\quad du = \cos x\,dx$. Then

$$\begin{aligned}\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx &= \int \frac{\left(1-u^2\right)du}{u^{1/2}} \\ &= \int \left(u^{-1/2} - u^{3/2}\right)du \\ &= 2u^{1/2} - \tfrac{2}{5}u^{5/2} + C \\ &= 2u^{1/2}\left(1 - \tfrac{1}{5}u^2\right) + C \\ &= 2\left(1 - \tfrac{1}{5}\sin^2 x\right)\sqrt{\sin x} + C\end{aligned}$$

17. Let $u = \sqrt{x}$ so that $x = u^2$ and $dx = 2u\,du$. Then

$$\begin{aligned}\int \frac{\cos^2\sqrt{x}}{\sqrt{x}}\,dx &= \int \frac{\cos^2 u}{u}2u\,du = \int 2\cos^2 u\,du \\ &= \int (1+\cos 2u)\,du \\ &= u + \tfrac{1}{2}\sin 2u + C \\ &= \sqrt{x} + \tfrac{1}{2}\sin\left(2\sqrt{x}\right) + C\end{aligned}$$

18. $$\begin{aligned}\int \frac{dx}{1-\sin x} &= \int \frac{1+\sin x}{1-\sin^2 x}\,dx = \int \frac{1+\sin x}{\cos^2 x}\,dx \\ &= \int \left(\sec^2 x + \sec x\tan x\right)dx \\ &= \tan x + \sec x + C\end{aligned}$$

19. Let $u = \sec x \quad\Rightarrow\quad du = \sec x\tan x\,dx$. Then

$$\begin{aligned}\int \tan x\sec^6 x\,dx &= \int \sec^5 x\sec x\tan x\,dx \\ &= \int u^5\,du = \tfrac{1}{6}u^6 + C \\ &= \tfrac{1}{6}\sec^6 x + C\end{aligned}$$

Or: Let $u = \tan x$, $du = \sec^2 x$. Then

$$\begin{aligned}\int \tan x\sec^6 x\,dx &= \int \tan x\left(1+\tan^2 x\right)^2\sec^2 x\,dx \\ &= \int \tan x\sec^2 x\,dx + 2\int \tan^3 x\sec^2 x\,dx \\ &\quad + \int \tan^5 x\sec^2 x\,dx \\ &= \tfrac{1}{2}\tan^2 x + \tfrac{1}{2}\tan^4 x + \tfrac{1}{6}\tan^6 x + C\end{aligned}$$

20. Let $u = \sec x \quad\Rightarrow\quad du = \sec x\tan x\,dx$. Then

$$\begin{aligned}\int \tan^3 x\sec^6 x\,dx &= \int \left(\sec^2 x - 1\right)\sec^5 x\sec x\tan x\,dx \\ &= \int \left(u^2-1\right)u^5\,du = \int \left(u^7 - u^5\right)du \\ &= \tfrac{1}{8}u^8 - \tfrac{1}{6}u^6 + C \\ &= \tfrac{1}{8}\sec^8 x - \tfrac{1}{6}\sec^6 x + C\end{aligned}$$

Or:

$$\begin{aligned}\int \tan^3 x\sec^6 x\,dx &= \int \tan^3 x\left(\tan^2 x + 1\right)^2\sec^2 x\,dx \\ &= \int v^3\left(v^2+1\right)^2\,dv \text{ (where } v = \tan x) \\ &= \int \left(v^7 + 2v^5 + v^3\right)dv \\ &= \tfrac{1}{8}v^8 + \tfrac{1}{3}v^6 + \tfrac{1}{4}v^4 + C_1 \\ &= \tfrac{1}{8}\tan^2 x + \tfrac{1}{3}\tan^6 x + \tfrac{1}{4}\tan^4 x + C_1\end{aligned}$$

21. $$\begin{aligned}\int \sec^4 x\,dx &= \int \left(\tan^2 x + 1\right)\sec^2 x\,dx \\ &= \int \tan^2 x\sec^2 x\,dx + \int \sec^2 x\,dx = \tfrac{1}{3}\tan^3 x + \tan x + C\end{aligned}$$

22. $$\begin{aligned}\int \sec^6 x\,dx &= \int \left(\tan^2 x + 1\right)^2\sec^2 x\,dx \\ &= \int \tan^4 x\sec^2 x\,dx + 2\int \tan^2 x\sec^2 x\,dx \\ &\quad + \int \sec^2 x\,dx \\ &= \tfrac{1}{5}\tan^5 x + \tfrac{2}{3}\tan^3 x + \tan x + C\end{aligned}$$

(Set $u = \tan x$ in the first two integrals.) Thus

$$\begin{aligned}\int_0^{\pi/4}\sec^6 x\,dx &= \left[\tfrac{1}{5}\tan^5 x + \tfrac{2}{3}\tan^3 x + \tan x\right]_0^{\pi/4} \\ &= \tfrac{1}{5} + \tfrac{2}{3} + 1 = \tfrac{28}{15}\end{aligned}$$

23. Let $u = \tan t \quad\Rightarrow\quad du = \sec^2 t\,dt$. Then $\int_0^{\pi/4}\tan^4 t\sec^2 t\,dt = \int_0^1 u^4\,du = \left[\tfrac{1}{5}u^5\right]_0^1 = \tfrac{1}{5}$.

24. Let $u = \tan x \quad\Rightarrow\quad du = \sec^2 x\,dx$. Then

$$\begin{aligned}\int_0^{\pi/4}\tan^2 x\sec^4 x\,dx &= \int_0^1 u^2\left(u^2+1\right)du \\ &= \int_0^1\left(u^4+u^2\right)du \\ &= \left[\tfrac{1}{5}u^5 + \tfrac{1}{3}u^3\right]_0^1 \\ &= \tfrac{1}{5} + \tfrac{1}{3} = \tfrac{8}{15}\end{aligned}$$

25. Let $u = \sec x \quad\Rightarrow\quad du = \sec x\tan x\,dx$. Then

$$\begin{aligned}\int \tan^3 x\sec^3 x\,dx &= \int \sec^2 x\tan^2 x\sec x\tan x\,dx \\ &= \int u^2\left(u^2-1\right)du = \int \left(u^4 - u^2\right)du \\ &= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C \\ &= \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

26. Let $u = \sec x \quad\Rightarrow\quad du = \sec x\tan x\,dx$. Then

$$\begin{aligned}\int_0^{\pi/3}\tan^5 x\sec x\,dx &= \int_0^{\pi/3}\left(\sec^2 x - 1\right)^2\sec x\tan x\,dx \\ &= \int_1^2\left(u^2-1\right)^2du \\ &= \int_1^2\left(u^4 - 2u^2 + 1\right)du \\ &= \left[\tfrac{1}{5}u^5 - \tfrac{2}{3}u^3 + u\right]_1^2 \\ &= \left(\tfrac{32}{5} - \tfrac{16}{3} + 2\right) - \left(\tfrac{1}{5} - \tfrac{2}{3} + 1\right) \\ &= \tfrac{38}{15}\end{aligned}$$

27. Let $u = \tan x \quad\Rightarrow\quad du = \sec^2 x\,dx$. Then

$$\begin{aligned}\int \frac{\sec^2 x}{\cot x}\,dx &= \int \tan x\sec^2 x\,dx = \int u\,du \\ &= \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\tan^2 x + C\end{aligned}$$

28. $$\begin{aligned}\int \cot^2 w\csc^4 w\,dw &= \int \cot^2 w\csc^2 w\csc^2 w\,dw \\ &= \int \cot^2 w\left(1+\cot^2 w\right)\csc^2 w\,dw \\ &= \int u^2\left(1+u^2\right)(-du)\quad [u = \cot w,\ du = -\csc^2 w\,dw] \\ &= -\int \left(u^2+u^4\right)du = -\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C \\ &= -\tfrac{1}{3}\cot^3 w - \tfrac{1}{5}\cot^5 w + C\end{aligned}$$

29. Let $u = \cot x \quad \Rightarrow \quad du = -\csc^2 x\,dx$. Then

$$\begin{aligned}\int \cot^3 x \csc^4 x\,dx &= \int \cot^3 x\left(\cot^2 x + 1\right)\csc^2 x\,dx \\ &= \int u^3\left(u^2+1\right)(-du) \\ &= -\tfrac{1}{6}u^6 - \tfrac{1}{4}u^4 + C \\ &= -\tfrac{1}{6}\cot^6 x - \tfrac{1}{4}\cot^4 x + C\end{aligned}$$

30. $\int e^x \cos^7(e^x)\,dx = \int \cos^7 u\,du \;\; [u = e^x,\, du = e^x\,dx]$

$$\begin{aligned}&= \int \left(\cos^2 u\right)^3 \cos u\,du = \int \left(1-\sin^2 u\right)^3 \cos u\,du \\ &= \int \left(1-v^2\right)^3 dv \;\; [v = \sin u,\, dv = \cos u\,du] \\ &= \int \left(1 - 3v^2 + 3v^4 - v^6\right) dv \\ &= v - v^3 + \tfrac{3}{5}v^5 - \tfrac{1}{7}v^7 + C \\ &= \sin u - \sin^3 u + \tfrac{3}{5}\sin^5 u - \tfrac{1}{7}\sin^7 u + C \\ &= \sin e^x - \sin^3 e^x + \tfrac{3}{5}\sin^5 e^x - \tfrac{1}{7}\sin^7 e^x + C\end{aligned}$$

31. $\displaystyle\int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{1-\sin^2 x}{\sin x}\,dx = \int (\csc x - \sin x)\,dx$

$$= \ln|\csc x - \cot x| + \cos x + C$$

32. $\displaystyle\int \frac{dx}{\sin^4 x} = \int \csc^4 x\,dx = \int \left(\cot^2 x + 1\right)\csc^2 x\,dx$

$$\begin{aligned}&= \int \left(u^2+1\right)(-du) \text{ (where } u = \cot x) \\ &= -\tfrac{1}{3}u^3 - u + C = -\tfrac{1}{3}\cot^3 x - \cot x + C\end{aligned}$$

33. $\int \cos 3x \cos 4x\,dx$

$$\begin{aligned}&= \int \tfrac{1}{2}\left[\cos(3x-4x) + \cos(3x+4x)\right]dx \\ &= \tfrac{1}{2}\int (\cos x + \cos 7x)\,dx \\ &= \tfrac{1}{2}\sin x + \tfrac{1}{14}\sin 7x + C\end{aligned}$$

34. $\int \sin 3x \sin 6x\,dx$

$$\begin{aligned}&= \int \tfrac{1}{2}\left[\cos(3x-6x) - \cos(3x+6x)\right]dx \\ &= \tfrac{1}{2}\int (\cos 3x - \cos 9x)\,dx \\ &= \tfrac{1}{6}\sin 3x - \tfrac{1}{18}\sin 9x + C\end{aligned}$$

35. $\int \sin x \cos 5x\,dx = \int \tfrac{1}{2}\left[\sin(x-5x) + \sin(x+5x)\right]dx$

$$\begin{aligned}&= \tfrac{1}{2}\int (-\sin 4x + \sin 6x)\,dx \\ &= \tfrac{1}{8}\cos 4x - \tfrac{1}{12}\cos 6x + C\end{aligned}$$

36. $\int \cos x \cos 2x \cos 3x\,dx$

$$\begin{aligned}&= \int \cos x \cdot \tfrac{1}{2}\left[\cos(2x-3x) + \cos(2x+3x)\right]dx \\ &= \tfrac{1}{2}\int \cos^2 x\,dx + \tfrac{1}{2}\int \cos x \cos 5x\,dx \\ &= \tfrac{1}{4}\int (1+\cos 2x)\,dx \\ &\quad + \tfrac{1}{2}\int \tfrac{1}{2}\left[\cos(x-5x) + \cos(x+5x)\right]dx \\ &= \tfrac{1}{4}x + \tfrac{1}{8}\sin 2x + \tfrac{1}{4}\int (\cos 4x + \cos 6x)\,dx \\ &= \tfrac{1}{4}x + \tfrac{1}{8}\sin 2x + \tfrac{1}{16}\sin 4x + \tfrac{1}{24}\sin 6x + C\end{aligned}$$

37. Let $x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and $\sqrt{1-x^2} = |\cos\theta| = \cos\theta$ (since $\cos\theta > 0$ for θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$). Thus

$$\begin{aligned}\int_{1/2}^{\sqrt{3}/2} \frac{dx}{x^2\sqrt{1-x^2}} &= \int_{\pi/6}^{\pi/3} \frac{\cos\theta\,d\theta}{\sin^2\theta\cos\theta} \\ &= \int_{\pi/6}^{\pi/3} \csc^2\theta\,d\theta = \left[-\cot\theta\right]_{\pi/6}^{\pi/3} \\ &= -\tfrac{1}{\sqrt{3}} - \left(-\sqrt{3}\right) \\ &= \tfrac{3}{\sqrt{3}} - \tfrac{1}{\sqrt{3}} = \tfrac{2}{\sqrt{3}}\end{aligned}$$

38. Let $x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$ and $\sqrt{4-x^2} = |2\cos\theta| = 2\cos\theta$, so

$$\begin{aligned}\int_0^2 x^3\sqrt{4-x^2}\,dx &= \int_0^{\pi/2} 8\sin^3\theta\,(2\cos\theta)\,(2\cos\theta)\;d\theta \\ &= 32\int_0^{\pi/2} \cos^2\theta\left(1-\cos^2\theta\right)\sin\theta\,d\theta \\ &= 32\int_1^0 u^2\left(1-u^2\right)(-du) \text{ (where } u = \cos\theta) \\ &= 32\int_0^1 \left(u^2-u^4\right)du = 32\left[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right]_0^1 \\ &= 32\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{64}{15}\end{aligned}$$

39. Let $u = 1 - x^2$. Then $du = -2x\,dx$, so

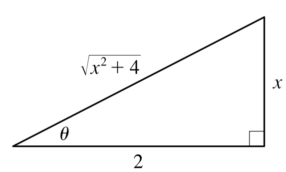
$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\,dx &= -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + C \\ &= -\sqrt{1-x^2} + C\end{aligned}$$

40. Let $u = 4 - x^2$. Then $du = -2x\,dx \quad \Rightarrow$

$$\begin{aligned}\int x\sqrt{4-x^2}\,dx &= -\tfrac{1}{2}\int \sqrt{u}\,du = -\tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2} + C \\ &= -\tfrac{1}{3}\left(4-x^2\right)^{3/2} + C\end{aligned}$$

41. Let $x = 2\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{x^2+4} = 2\sec\theta$, so

$$\begin{aligned}\int_0^2 \frac{x^3}{\sqrt{x^2+4}}\,dx &= \int_0^{\pi/4} \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta \\ &= 8\int_0^{\pi/4}\left(\sec^2\theta - 1\right)\sec\theta\tan\theta\,d\theta \\ &= 8\left[\tfrac{1}{3}\sec^3\theta - \sec\theta\right]_0^{\pi/4} \\ &= 8\left[\tfrac{1}{3}\cdot 2\sqrt{2} - \sqrt{2}\right] - 8\left[\tfrac{1}{3} - 1\right] \\ &= \tfrac{8}{3}\left(2-\sqrt{2}\right)\end{aligned}$$



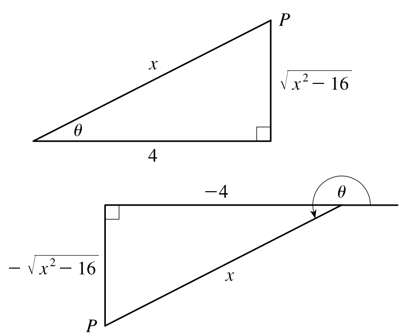
42. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\, d\theta$ and $\sqrt{9+x^2} = 3\sec\theta$.

$$\int_0^3 \frac{dx}{\sqrt{9+x^2}} = \int_0^{\pi/4} \frac{3\sec^2\theta\, d\theta}{3\sec\theta} = \int_0^{\pi/4} \sec\theta\, d\theta$$
$$= [\ln|\sec\theta + \tan\theta|]_0^{\pi/4}$$
$$= \ln\left(\sqrt{2}+1\right) - \ln 1 = \ln\left(\sqrt{2}+1\right)$$

43. Let $x = 4\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 4\sec\theta\tan\theta\, d\theta$ and $\sqrt{x^2-16} = 4|\tan\theta| = 4\tan\theta$. Thus

$$\int \frac{dx}{x^3\sqrt{x^2-16}} = \int \frac{4\sec\theta\tan\theta\, d\theta}{64\sec^3\theta \cdot 4\tan\theta}$$
$$= \frac{1}{64}\int \cos^2\theta\, d\theta = \frac{1}{128}\int (1+\cos 2\theta)\, d\theta$$
$$= \frac{1}{128}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{128}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{128}\left(\sec^{-1}\frac{x}{4} + \frac{4\sqrt{x^2-16}}{x^2}\right) + C$$

by the diagrams for $0 \le \theta < \frac{\pi}{2}$ and $\pi \le \theta < \frac{3\pi}{2}$, where the labels of the legs in the second diagram indicate the x-and y-coordinates of P rather than the lengths of those sides. Henceforth we omit the second diagram from our solutions.

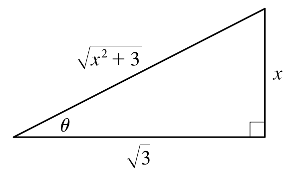


44. Let $u = x^2 + 4 \quad\Rightarrow\quad du = 2x\,dx$. Then

$$\int \frac{x\,dx}{(x^2+4)^{5/2}} = \frac{1}{2}\int u^{-5/2}du = \frac{1}{2}\left(-\frac{2}{3}\right)u^{-3/2} + C$$
$$= \left(-\frac{1}{3}\right)u^{-3/2} + C$$
$$= -\frac{1}{3}\left(x^2+4\right)^{-3/2} + C$$

45. Let $x = \sqrt{3}\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\int \frac{dx}{x\sqrt{x^2+3}} = \int \frac{\sqrt{3}\sec^2\theta d\theta}{\sqrt{3}\tan\theta\sqrt{3}\sec\theta}$$
$$= \frac{1}{\sqrt{3}}\int \csc\theta d\theta$$
$$= \frac{1}{\sqrt{3}}\ln|\csc\theta - \cot\theta| + C$$
$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+3}-\sqrt{3}}{x}\right| + C$$

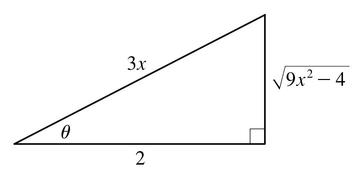


46. Let $u = 25 + x^2$, so $du = 2x\,dx$. Then

$$\int x\sqrt{25+x^2}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$
$$= \frac{1}{3}\left(x^2+25\right)^{3/2} + C$$

47. $9x^2 - 4 = (3x)^2 - 4$, so let $3x = 2\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \frac{2}{3}\sec\theta\tan\theta\, d\theta$ and $\sqrt{9x^2-4} = 2\tan\theta$.

$$\int \frac{\sqrt{9x^2-4}}{x}dx = \int \frac{2\tan\theta}{\frac{2}{3}\sec\theta}\cdot\frac{2}{3}\sec\theta\tan\theta d\theta$$
$$= 2\int \tan^2\theta d\theta = 2\int \left(\sec^2\theta - 1\right)d\theta$$
$$= (2\tan\theta - \theta) + C$$
$$= \sqrt{9x^2-4} - 2\sec^{-1}\left(\frac{3x}{2}\right) + C$$



48. Let $u = 9 - x^2$, so $du = -2x\,dx$. Then

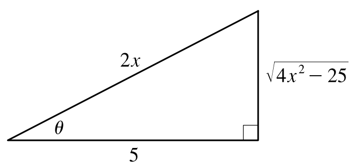
$$\int_0^3 x\sqrt{9-x^2}\,dx = -\frac{1}{2}\int_9^0 \sqrt{u}\,du = -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_9^0$$
$$= -\frac{1}{3}(0-27) = 9$$

49. Let $u = 1 + x^2$, $du = 2x\,dx$. Then

$$\int 5x\sqrt{1+x^2}\,dx = \frac{5}{2}\int u^{1/2}\,du = \frac{5}{3}u^{3/2} + C$$
$$= \frac{5}{3}\left(1+x^2\right)^{3/2} + C$$

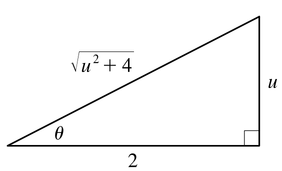
50. Let $2x = 5\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$$\begin{aligned}\int \frac{dx}{(4x^2-25)^{3/2}} &= \int \frac{\frac{5}{2}\sec\theta\tan\theta\,d\theta}{125\tan^3\theta} \\ &= \frac{1}{50}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \\ &= -\frac{1}{50\sin\theta} + C \quad (\text{set } u = \sin\theta) \\ &= -\frac{1}{50}\cdot\frac{2x}{\sqrt{4x^2-25}} + C \\ &= -\frac{x}{25\sqrt{4x^2-25}} + C\end{aligned}$$



51. $x^2 + 4x + 8 = (x+2)^2 + 4$. Let $u = x + 2 \Rightarrow du = dx$. Then let $u = 2\tan\theta$.

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+4x+8}} &= \int \frac{du}{\sqrt{u^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} \\ &= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 \\ &= \ln\left(\frac{\sqrt{u^2+4}+u}{2}\right) + C_1 \\ &= \ln\left(\sqrt{u^2+4}+u\right) + C \\ &= \ln\left(\sqrt{x^2+4x+8}+x+2\right) + C\end{aligned}$$



52. $2x - x^2 = -\left(x^2 - 2x + 1\right) + 1 = 1 - (x-1)^2$. Let $u = x - 1$. Then $du = dx$ and

$$\begin{aligned}\int \sqrt{2x-x^2}\,dx &= \int \sqrt{1-u^2}\,du \\ &= \int \cos^2\theta\,d\theta \quad (u = \sin\theta, -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}) \\ &= \tfrac{1}{2}\int (1 + \cos 2\theta)\,d\theta = \tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C \\ &= \tfrac{1}{2}\left(\sin^{-1}u + u\sqrt{1-u^2}\right) + C \\ &= \tfrac{1}{2}\left[\sin^{-1}(x-1) + (x-1)\sqrt{2x-x^2}\right] + C\end{aligned}$$

53. Let $u = e^t \Rightarrow du = e^t dt$. Then

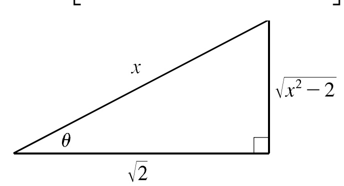
$$\begin{aligned}\int e^t\sqrt{9-e^{2t}}\,dt &= \int \sqrt{9-u^2}\,du \\ &= \int (3\cos\theta)\,3\cos\theta\,d\theta \quad (u = 3\sin\theta, -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}) \\ &= 9\int \cos^2\theta\,d\theta = \tfrac{9}{2}\int (1+\cos 2\theta)\,d\theta \\ &= \tfrac{9}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{9}{2}\left[\sin^{-1}\left(\frac{u}{3}\right) + \frac{u}{3}\cdot\frac{\sqrt{9-u^2}}{3}\right] + C \\ &= \tfrac{9}{2}\sin^{-1}\left(\tfrac{1}{3}e^t\right) + \tfrac{1}{2}e^t\sqrt{9-e^{2t}} + C\end{aligned}$$

54. Let $u = e^t$. Then $t = \ln u$ and $dt = du/u$. Hence $I = \int \sqrt{e^{2t}-9}\,dt = \int \left(\sqrt{u^2-9}/u\right)du$. Now let $u = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $\sqrt{u^2-9} = 3\tan\theta$ and $du = 3\sec\theta\tan\theta\,d\theta$, so

$$\begin{aligned}I &= \int \frac{3\tan\theta}{3\sec\theta}3\sec\theta\tan\theta\,d\theta = 3\int \tan^2\theta\,d\theta \\ &= 3\int \left(\sec^2\theta - 1\right)d\theta = 3(\tan\theta - \theta) + C \\ &= 3\left[\tfrac{1}{3}\sqrt{u^2-9} - \sec^{-1}\left(\tfrac{1}{3}u\right)\right] + C \\ &= \sqrt{e^{2t}-9} - 3\sec^{-1}\left(\tfrac{1}{3}e^t\right) + C\end{aligned}$$

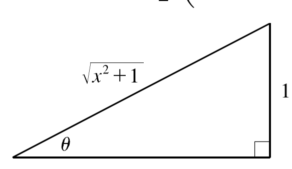
55. Let $x = \sqrt{2}\sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$. Then

$$\begin{aligned}\int \frac{dx}{x^4\sqrt{x^2-2}} &= \int \frac{\sqrt{2}\sec\theta\tan\theta\,d\theta}{4\sec^4\theta\sqrt{2}\tan\theta} = \tfrac{1}{4}\int \cos^3\theta\,d\theta \\ &= \tfrac{1}{4}\int \left(1-\sin^2\theta\right)\cos\theta\,d\theta \\ &= \tfrac{1}{4}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta\right) + C \text{ (where } u = \sin\theta) \\ &= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{\left(x^2-2\right)^{3/2}}{3x^3}\right] + C\end{aligned}$$



56. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\begin{aligned}\int \frac{dx}{(1-x^2)^2} &= \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int \cos^2\theta\,d\theta \\ &= \tfrac{1}{2}\int (1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C \\ &= \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C = \frac{1}{2}\left(\tan^{-1}x + \frac{x}{x^2+1}\right) + C\end{aligned}$$



57. $f(x) = \left(4 - x^2\right)^{3/2}$ on $[0, 2]$ $\Rightarrow$

$f_{\text{ave}} = \frac{1}{2-0} \int_0^2 \left(4 - x^2\right)^{3/2} dx$. Let $x = 2\sin\theta$ $\Rightarrow$

$dx = 2\cos\theta\, d\theta$ and $\left(4 - x^2\right)^{3/2} = (2\cos\theta)^3$. So

$$f_{\text{ave}} = \tfrac{1}{2} \int_0^{\pi/2} (2\cos\theta)^3\, 2\cos\theta\, d\theta = 8 \int_0^{\pi/2} \cos^4\theta\, d\theta$$
$$= 8\left[\tfrac{3}{8}\theta + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{32}\sin 4\theta\right]_0^{\pi/2}$$

[by Archived Problem 6.2.7]

$$= 8\left[\left(\tfrac{3\pi}{16} + 0 + 0\right) - (0 + 0 + 0)\right] = \tfrac{3\pi}{2}$$