6.3 PARTIAL FRACTIONS

A Click here for answers.

S Click here for solutions.

1–18 ■ Write out the form of the partial fraction decomposition of the function (as in Example 6). Do not determine the numerical values of the coefficients.

1. $\dfrac{3}{(2x+3)(x-1)}$

2. $\dfrac{5}{2x^2-3x-2}$

3. $\dfrac{x^2+9x-12}{(3x-1)(x+6)^2}$

4. $\dfrac{z^2-4z}{(3z+5)^3(z+2)}$

5. $\dfrac{1}{x^4-x^3}$

6. $\dfrac{x^4+x^3-x^2-x+1}{x^3-x}$

7. $\dfrac{x^2+1}{x^2-1}$

8. $\dfrac{x^3-4x^2+2}{(x^2+1)(x^2+2)}$

9. $\dfrac{x+1}{x^2+2x}$

10. $\dfrac{7}{2x^2+5x-12}$

11. $\dfrac{1}{(x-1)(x+2)}$

12. $\dfrac{x^2+3x-4}{(2x-1)^2(2x+3)}$

13. $\dfrac{x^3-x^2}{(x-6)(5x+3)^3}$

14. $\dfrac{1}{x^6-x^3}$

15. $\dfrac{19x}{(x-1)^3(4x^2+5x+3)^2}$

16. $\dfrac{x^3+x^2+1}{x^4+x^3+2x^2}$

17. $\dfrac{3-11x}{(x-2)^3(x^2+1)(2x^2+5x+7)^2}$

18. $\dfrac{x^4}{(x^2+9)^3}$

19–51 ■ Evaluate the integral.

19. $\displaystyle\int \frac{x^2}{x+1}\,dx$

20. $\displaystyle\int \frac{y}{y+2}\,dy$

21. $\displaystyle\int \frac{x^2+1}{x^2-x}\,dx$

22. $\displaystyle\int_0^2 \frac{x^3+x^2-12x+1}{x^2+x-12}\,dx$

23. $\displaystyle\int_2^3 \frac{1}{x^3+x^2-2x}\,dx$

24. $\displaystyle\int_2^4 \frac{4x-1}{(x-1)(x+2)}\,dx$

25. $\displaystyle\int_3^7 \frac{1}{(x+1)(x-2)}\,dx$

26. $\displaystyle\int \frac{6x-5}{2x+3}\,dx$

27. $\displaystyle\int \frac{1}{x(x+1)(2x+3)}\,dx$

28. $\displaystyle\int_2^3 \frac{6x^2+5x-3}{x^3+2x^2-3x}\,dx$

29. $\displaystyle\int_0^1 \frac{x}{x^2+4x+4}\,dx$

30. $\displaystyle\int \frac{18-2x-4x^2}{x^3+4x^2+x-6}\,dx$

31. $\displaystyle\int \frac{dx}{(x-1)(x-2)(x-3)}$

32. $\displaystyle\int \frac{dx}{x(x+1)(2x+3)}$

33. $\displaystyle\int \frac{3x^2-6x+2}{2x^3-3x^2+x}\,dx$

34. $\displaystyle\int_0^1 \frac{x^3}{x^2+1}\,dx$

35. $\displaystyle\int_1^2 \frac{x^2+3}{x^3+2x}\,dx$

36. $\displaystyle\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx$

37. $\displaystyle\int \frac{2t^3-t^2+3t-1}{(t^2+1)(t^2+2)}\,dt$

38. $\displaystyle\int \frac{x^4}{x^4-1}\,dx$

39. $\displaystyle\int \frac{2x^3-x}{x^4-x^2+1}\,dx$

40. $\displaystyle\int_0^1 \frac{x-1}{x^2+2x+2}\,dx$

41. $\displaystyle\int_0^1 \frac{x}{x^2+x+1}\,dx$

42. $\displaystyle\int_{-1/2}^{1/2} \frac{4x^2+5x+7}{4x^2+4x+5}\,dx$

43. $\displaystyle\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}\,dx$

44. $\displaystyle\int \frac{(2\sin x-3)\cos x}{\sin^2 x-3\sin x+2}\,dx$

45. $\displaystyle\int \frac{\sin x\cos^2 x}{5+\cos^2 x}\,dx$

46. $\displaystyle\int \frac{x^2+7x-6}{(x+1)(x^2-4x+7)}\,dx$

47. $\displaystyle\int \frac{4x+1}{(x-3)(x^2+6x+12)}\,dx$

48. $\displaystyle\int \frac{x+1}{(x^2+x+2)}\,dx$

49. $\displaystyle\int \frac{3x^4-2x^3+20x^2-5x+34}{(x-1)(x^2+4)^2}\,dx$

50. $\displaystyle\int \frac{8x}{(x^2+4)^3}\,dx$

51. $\displaystyle\int \frac{x^2+1}{(x^3+3x)^2}\,dx$

6.3 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{A}{2x+3}+\frac{B}{x-1}$

2. $\frac{A}{2x+1}+\frac{B}{x-2}$

3. $\frac{A}{3x-1}+\frac{B}{x+6}+\frac{C}{(x+6)^2}$

4. $\frac{A}{3z+5}+\frac{B}{(3z+5)^2}+\frac{C}{(3z+5)^3}+\frac{D}{z+2}$

5. $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}$

6. $x+1+\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$

7. $1+\frac{A}{x-1}+\frac{B}{x+1}$

8. $\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2}$

9. $\frac{A}{x}+\frac{B}{x+2}$

10. $\frac{A}{2x-3}+\frac{B}{x+4}$

11. $\frac{A}{x-1}+\frac{B}{x+2}$

12. $\frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{C}{2x+3}$

13. $\frac{A}{x-6}+\frac{B}{5x+3}+\frac{C}{(5x+3)^2}+\frac{D}{(5x+3)^3}$

14. $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}+\frac{Ex+F}{x^2+x+1}$

15. $\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{Dx+E}{4x^2+5x+3}+\frac{Fx+G}{(4x^2+5x+3)^2}$

16. $\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+x+2}$

17. $\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{2x^2+5x+7}+\frac{Hx+I}{(2x^2+5x+7)^2}$

18. $\frac{Ax+B}{x^2+9}+\frac{Cx+D}{(x^2+9)^2}+\frac{Ex+F}{(x^2+9)^3}$

19. $\frac{1}{2}x^2-x+\ln|x+1|+C$

20. $y-2\ln|y+2|+C$

21. $x+\ln\frac{(x-1)^2}{|x|}+C$

22. $2+\frac{1}{7}\ln\frac{2}{9}$

23. $\frac{1}{2}\ln 2-\frac{1}{2}\ln 3+\frac{1}{6}\ln 5$

24. $\ln\frac{81}{8}$

25. $\frac{1}{3}\ln\frac{5}{2}$

26. $3x-7\ln|2x+3|+C$

27. $\frac{1}{3}\ln|x|-\ln|x+1|+\frac{2}{3}\ln|2x+3|+C$

28. $4\ln 6-3\ln 5$

29. $\ln\frac{3}{2}-\frac{1}{3}$

30. $\ln|x-1|-2\ln|x+2|-3\ln|x+3|+C$

31. $\frac{1}{2}\ln\frac{|(x-1)(x-3)|}{(x-2)^2}+C$

32. $\frac{1}{3}\ln|x|-\ln|x+1|+\frac{2}{3}\ln|2x+3|+C$

33. $2\ln|x|-\ln|x-1|+\frac{1}{2}\ln|2x-1|+C$

34. $\frac{1}{2}(1-\ln 2)$

35. $\frac{5}{4}\ln 2$

36. $(x-1)^2+\ln\sqrt{x^2+1}-3\tan^{-1}x+C$

37. $\frac{1}{2}\ln(t^2+1)+\frac{1}{2}\ln(t^2+2)-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}t\right)+C$

38. $x+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$

39. $\frac{1}{2}\ln(x^4-x^2+1)+C$

40. $\frac{1}{2}\ln\frac{5}{2}-2\tan^{-1}2+\frac{\pi}{2}$

41. $\ln\sqrt{3}-\frac{\pi}{6\sqrt{3}}$

42. $1+\frac{1}{8}\ln 2+\frac{3\pi}{32}$

43. $\frac{3}{2}\ln(x^2+1)-3\tan^{-1}x+\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}x\right)+C$

44. $\ln\left|\sin^2 x-3\sin x+2\right|+C$

45. $-\cos x+\sqrt{5}\tan^{-1}\left(\frac{1}{\sqrt{5}}\cos x\right)+C$

46. $-\ln|x+1|+\ln\left(x^2-4x+7\right)+\dfrac{5}{\sqrt{3}}\tan^{-1}\left(\dfrac{x-2}{\sqrt{3}}\right)+C$

47. $\frac{1}{3}\ln|x-3|-\frac{1}{6}\ln\left(x^2+6x+12\right)+\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{x+3}{\sqrt{3}}\right)+C$

48. $\dfrac{x-3}{7\left(x^2+x+2\right)}+\dfrac{2\sqrt{7}}{49}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{7}}\right)+C$

49. $2\ln|x-1|+\frac{1}{2}\ln\left(x^2+4\right)-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}x\right)$

$$+\frac{1}{2\left(x^2+4\right)}+\frac{1}{8}\left[\tan^{-1}\left(\tfrac{1}{2}x\right)+\frac{2x}{x^2+4}\right]+C$$

50. $-\dfrac{2}{\left(x^2+4\right)^2}+C$

51. $-\dfrac{1}{3\left(x^3+3x\right)}+C$

6.3 SOLUTIONS

E Click here for exercises.

1. $\dfrac{3}{(2x+3)(x-1)}=\dfrac{A}{2x+3}+\dfrac{B}{x-1}$

2. $\dfrac{5}{2x^2-3x-2}=\dfrac{5}{(2x+1)(x-2)}=\dfrac{A}{2x+1}+\dfrac{B}{x-2}$

3. $\dfrac{x^2+9x-12}{(3x-1)(x+6)^2}=\dfrac{A}{3x-1}+\dfrac{B}{x+6}+\dfrac{C}{(x+6)^2}$

4. $\dfrac{z^2-4z}{(3z+5)^3(z+2)}=\dfrac{A}{3z+5}+\dfrac{B}{(3z+5)^2}+\dfrac{C}{(3z+5)^3}+\dfrac{D}{z+2}$

5. $\dfrac{1}{x^4-x^3}=\dfrac{1}{x^3(x-1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-1}$

6. $\dfrac{x^4+x^3-x^2-x+1}{x^3-x}=x+1+\dfrac{1}{x(x+1)(x-1)}$
$=x+1+\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-1}$

7. $\dfrac{x^2+1}{x^2-1}=1+\dfrac{2}{(x-1)(x+1)}=1+\dfrac{A}{x-1}+\dfrac{B}{x+1}$

8. $\dfrac{x^3-4x^2+2}{(x^2+1)(x^2+2)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+2}$

9. $\dfrac{x+1}{x(x+2)}=\dfrac{A}{x}+\dfrac{B}{x+2}$

10. $\dfrac{7}{(2x-3)(x+4)}=\dfrac{A}{2x-3}+\dfrac{B}{x+4}$

11. $\dfrac{1}{(x-1)(x+2)}=\dfrac{A}{x-1}+\dfrac{B}{x+2}$

12. $\dfrac{x^2+3x-4}{(2x-1)^2(2x+3)}=\dfrac{A}{2x-1}+\dfrac{B}{(2x-1)^2}+\dfrac{C}{2x+3}$

13. $\dfrac{x^3-x^2}{(x-6)(5x+3)^3}=\dfrac{A}{x-6}+\dfrac{B}{5x+3}+\dfrac{C}{(5x+3)^2}+\dfrac{D}{(5x+3)^3}$

14. $\dfrac{1}{x^6-x^3}=\dfrac{1}{x^3(x^3-1)}=\dfrac{1}{x^3(x-1)(x^2+x+1)}$
$=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-1}+\dfrac{Ex+F}{x^2+x+1}$

15. $\dfrac{19x}{(x-1)^3(4x^2+5x+3)^2}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}+\dfrac{Dx+E}{4x^2+5x+3}+\dfrac{Fx+G}{(4x^2+5x+3)^2}$

16. $\dfrac{x^3+x^2+1}{x^4+x^3+2x^2}=\dfrac{x^3+x^2+1}{x^2(x^2+x+2)}$
$=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+x+2}$

17. $\dfrac{3-11x}{(x-2)^3(x^2+1)(2x^2+5x+7)^2}$
$=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{(x-2)^3}+\dfrac{Dx+E}{x^2+1}+\dfrac{Fx+G}{2x^2+5x+7}+\dfrac{Hx+I}{(2x^2+5x+7)^2}$

18. $\dfrac{x^4}{(x^2+9)^3}=\dfrac{Ax+B}{x^2+9}+\dfrac{Cx+D}{(x^2+9)^2}+\dfrac{Ex+F}{(x^2+9)^3}$

19. $\displaystyle\int\frac{x^2}{x+1}\,dx=\int\left(x-1+\frac{1}{x+1}\right)dx$
$=\frac{1}{2}x^2-x+\ln|x+1|+C$

20. $\displaystyle\int\frac{y}{y+2}\,dy=\int\left(1-\frac{2}{y+2}\right)dy=y-2\ln|y+2|+C$

21. $\dfrac{x^2+1}{x^2-x}=1+\dfrac{x+1}{x(x-1)}=1-\dfrac{1}{x}+\dfrac{2}{x-1}$, so
$\displaystyle\int\frac{x^2+1}{x^2-x}\,dx=x-\ln|x|+2\ln|x-1|+C$
$\displaystyle=x+\ln\frac{(x-1)^2}{|x|}+C$

22. $\dfrac{x^3+x^2-12x+1}{x^2+x-12}=x+\dfrac{1}{x^2+x-12}$
$=x+\dfrac{1}{(x-3)(x+4)}=x+\dfrac{1}{7}\left(\dfrac{1}{x-3}-\dfrac{1}{x+4}\right)$
So $\displaystyle\int_0^2\frac{x^3+x^2-12x+1}{x^2+x-12}\,dx$
$=\left[\frac{1}{2}x^2+\frac{1}{7}(\ln|x-3|-\ln|x+4|)\right]_0^2$
$=2+\frac{1}{7}\ln\frac{2}{9}$

23. $\dfrac{1}{x^3+x^2-2x} = \dfrac{1}{x(x+2)(x-1)}$

$= \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-1}$

$\Rightarrow \quad 1 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$.

Setting $x = 0$ gives $1 = -2A$, so $A = -\frac{1}{2}$. Setting $x = -2$ gives $1 = 6B$, so $B = \frac{1}{6}$. Setting $x = 1$ gives $1 = 3C$, so $C = \frac{1}{3}$. Now

$$\int_2^3 \frac{1}{x^3+x^2-2x}\,dx$$
$$= \int_2^3 \left(\frac{-1/2}{x} + \frac{1/6}{x+2} + \frac{1/3}{x-1} \right) dx$$
$$= \left[-\tfrac{1}{2}\ln|x| + \tfrac{1}{6}\ln|x+2| + \tfrac{1}{3}\ln|x-1| \right]_2^3$$
$$= -\tfrac{1}{2}\ln 3 + \tfrac{1}{6}\ln 5 + \tfrac{1}{3}\ln 2 + \tfrac{1}{2}\ln 2 - \tfrac{1}{6}\ln 4 - \tfrac{1}{3}\ln 1$$
$$= \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\ln 3 + \tfrac{1}{6}\ln 5$$

24. $\dfrac{4x-1}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} \quad \Rightarrow$

$4x - 1 = A(x+2) + B(x-1)$ Take $x = 1$ to get $3 = 3A$, then $x = -2$ to get $-9 = -3B \quad \Rightarrow \quad A = 1$, $B = 3$. Now

$$\int_2^4 \frac{4x-1}{(x-1)(x+2)}\,dx = \int_2^4 \left[\frac{1}{x-1} + \frac{3}{x+2} \right] dx$$
$$= \left[\ln(x-1) + 3\ln(x+2) \right]_2^4$$
$$= \ln 3 + 3\ln 6 - \ln 1 - 3\ln 4$$
$$= 4\ln 3 - 3\ln 2 = \ln \tfrac{81}{8}$$

25. $\dfrac{1}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2} \quad \Rightarrow$

$1 = A(x-2) + B(x+1)$. Taking $x = -1$, then $x = 2$, gives $A = -\frac{1}{3}$, $B = \frac{1}{3}$. Hence

$$\int_3^7 \frac{dx}{(x+1)(x-2)} = \frac{1}{3}\int_3^7 \left[\frac{1}{x-2} - \frac{1}{x+1} \right] dx$$
$$= \tfrac{1}{3}\left[\ln|x-2| - \ln|x+1| \right]_3^7$$
$$= \tfrac{1}{3}(\ln 5 - \ln 8 - \ln 1 + \ln 4)$$
$$= \tfrac{1}{3}\ln \tfrac{5}{2}$$

26. $\displaystyle\int \frac{6x-5}{2x+3}\,dx = \int \left[3 - \frac{14}{2x+3} \right] dx$

$= 3x - 7\ln|2x+3| + C$

27. $\dfrac{1}{x(x+1)(2x+3)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{2x+3} \quad \Rightarrow$

$1 = A(x+1)(2x+3) + B(x)(2x+3) + C(x)(x+1)$.

Set $x = 0$ to get $A = \frac{1}{3}$, take $x = -1$ to get $B = -1$, and finally set $x = -\frac{3}{2}$, giving $C = \frac{4}{3}$. Now

$$\int \frac{dx}{x(x+1)(2x+3)} = \int \left[\frac{1/3}{x} - \frac{1}{x+1} + \frac{4/3}{2x+3} \right] dx$$
$$= \tfrac{1}{3}\ln|x| - \ln|x+1| + \tfrac{2}{3}\ln|2x+3| + C$$

28. $\dfrac{6x^2+5x-3}{x^3+2x^2-3x} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-1} \quad \Rightarrow$

$6x^2 + 5x - 3$

$= A(x+3)(x-1) + B(x)(x-1) + C(x)(x+3)$

Set $x = 0$ to get $A = 1$, then take $x = -3$ to get $B = 3$, then set $x = 1$ to get $C = 2$:

$$\int_2^3 \frac{6x^2+5x-3}{x^3+2x^2-3x}\,dx = \int_2^3 \left[\frac{1}{x} + \frac{3}{x+3} + \frac{2}{x-1} \right] dx$$
$$= \left[\ln x + 3\ln(x+3) + 2\ln(x-1) \right]_2^3$$
$$= (\ln 3 + 3\ln 6 + 2\ln 2) - (\ln 2 + 3\ln 5)$$
$$= 4\ln 6 - 3\ln 5$$

29. $\dfrac{x}{x^2+4x+4} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} \quad \Rightarrow$

$x = A(x+2) + B$. Set $x = -2$ to get $B = -2$ and equate coefficients of x to get $A = 1$. Then

$$\int_0^1 \frac{x\,dx}{x^2+4x+4} = \int_0^1 \left[\frac{1}{x+2} - \frac{2}{(x+2)^2} \right] dx$$
$$= \left[\ln(x+2) + \frac{2}{x+2} \right]_0^1$$
$$= \ln 3 + \tfrac{2}{3} - (\ln 2 + 1) = \ln \tfrac{3}{2} - \tfrac{1}{3}$$

30. $\dfrac{18-2x-4x^2}{x^3+4x^2+x-6} = \dfrac{18-2x-4x^2}{(x-1)(x+2)(x+3)}$

$= \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x+3} \quad \Rightarrow$

$18 - 2x - 4x^2 =$

$A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2)$.

Set $x = 1$ to get $A = 1$. Now setting $x = -2$ gives $B = -2$, and setting $x = -3$ gives $C = -3$. Then

$$\int \frac{18-2x-4x^2}{x^3+4x^2+x-6}\,dx$$
$$= \int \left(\frac{1}{x-1} - \frac{2}{x+2} - \frac{3}{x+3} \right) dx$$
$$= \ln|x-1| - 2\ln|x+2| - 3\ln|x+3| + C$$

31. $\dfrac{1}{(x-1)(x-2)9x-3} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3} \quad \Rightarrow$

$1 = A(x-2)(x-3) + B(x-1)(x-3)$

$+ C(x-1)(x-2)$

Set $x = 1$ to get $A = \frac{1}{2}$, and take $x = 2$ to get $B = -1$. Finally, setting $x = 3$ gives $C = \frac{1}{2}$. Now

$$\int \frac{dx}{(x-1)(x-2)(x-3)}$$
$$= \int \left(\frac{1/2}{x-1} - \frac{1}{x-2} + \frac{1/2}{x-3} \right) dx$$
$$= \tfrac{1}{2}\ln|x-1| - \ln|x-2| + \tfrac{1}{2}\ln|x-3| + C$$
$$= \frac{1}{2}\ln\frac{|(x-1)(x-3)|}{(x-2)^2} + C$$

32. $\dfrac{1}{x(x+1)(2x+3)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{2x+3} \quad \Rightarrow$
$1 = A(x+1)(2x+3) + B(x)(2x+3) + C(x)(x+1)$.
Set $x=0$ to get $A = \frac{1}{3}$. Take $x = -1$ to get $B = -1$, and finally setting $x = -\frac{3}{2}$ gives $C = \frac{4}{3}$. Now

$$\int \frac{dx}{x(x+1)(2x+3)} = \int \left(\frac{1/3}{x} - \frac{1}{x+1} + \frac{4/3}{2x+3} \right) dx = \tfrac{1}{3}\ln|x| - \ln|x+1| + \tfrac{2}{3}\ln|2x+3| + C$$

33. $\dfrac{3x^2-6x+2}{2x^3-3x^2+x} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{2x-1} \quad \Rightarrow$
$3x^2-6x+2 = A(x-1)(2x-1) + B(x)(2x-1) + C(x)(x-1)$
Setting $x=1$ gives $B=-1$. Set $x = \frac{1}{2}$ to get $C = 1$, and set $x=0$ to get $A=2$. Then

$$\int \frac{3x^2-6x+2}{2x^3-3x^2+x}\,dx = \int \left(\frac{2}{x} - \frac{1}{x-1} + \frac{1}{2x-1} \right) dx = 2\ln|x| - \ln|x-1| + \tfrac{1}{2}\ln|2x-1| + C$$

34. $\dfrac{x^3}{x^2+1} = \dfrac{(x^3+x)-x}{x^2+1} = x - \dfrac{x}{x^2+1}$, so

$$\int_0^1 \frac{x^3}{x^2+1}\,dx = \int_0^1 x\,dx - \int_0^1 \frac{x\,dx}{x^2+1} = \left[\tfrac{1}{2}x^2\right]_0^1 - \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du \text{ (where } u = x^2+1,\ du = 2x\,dx) = \tfrac{1}{2} - \left[\tfrac{1}{2}\ln u\right]_1^2 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2 = \tfrac{1}{2}(1-\ln 2)$$

35. $\dfrac{x^2+3}{x^3+2x} = \dfrac{x^2+3}{x(x^2+2)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2} \quad \Rightarrow$
$x^2+3 = A(x^2+2) + (Bx+C)x = (A+B)x^2 + Cx + 2A \quad \Rightarrow$
$A+B=1$, $C=0$, and $2A=3 \quad \Rightarrow \quad A = \frac{3}{2}$, $B = -\frac{1}{2}$, and $C=0$. Now

$$\int_1^2 \frac{x^2+3}{x^3+2x}\,dx = \int_1^2 \left(\frac{3/2}{x} - \frac{x/2}{x^2+2} \right) dx = \left[\tfrac{3}{2}\ln x - \tfrac{1}{4}\ln(x^2+2)\right]_1^2 = \tfrac{3}{2}\ln 2 - \tfrac{1}{4}\ln 6 - \tfrac{3}{2}\ln 1 + \tfrac{1}{4}\ln 3 = \tfrac{3}{2}\ln 2 - \tfrac{1}{4}\ln 2 - \tfrac{1}{4}\ln 3 - 0 + \tfrac{1}{4}\ln 3 = \left(\tfrac{3}{2} - \tfrac{1}{4}\right)\ln 2 = \tfrac{5}{4}\ln 2$$

36. $\dfrac{3x^2-4x+5}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1} \quad \Rightarrow$
$3x^2-4x+5 = A(x^2+1) + (Bx+C)(x-1)$. Take $x=1$ to get $4 = 2A$ or $A=2$. Now
$(Bx+C)(x-1) = 3x^2-4x+5-2(x^2+1) = x^2-4x+3$
Equating coefficients of x^2 and then comparing the constant terms, we get $B=1$ and $C=-3$. Hence,

$$\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx = \int \left[\frac{2}{x-1} + \frac{x-3}{x^2+1}\right] dx = 2\ln|x-1| + \int \frac{x\,dx}{x^2+1} - 3\int \frac{dx}{x^2+1} = 2\ln|x-1| + \tfrac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + C = \ln(x-1)^2 + \ln\sqrt{x^2+1} - 3\tan^{-1}x + C$$

37. $\dfrac{2t^3-t^2+3t-1}{(t^2+1)(t^2+2)} = \dfrac{At+B}{t^2+1} + \dfrac{Ct+D}{t^2+2} \quad \Rightarrow$
$2t^3-t^2+3t-1 = (At+B)(t^2+2) + (Ct+D)(t^2+1) = (A+C)t^3 + (B+D)t^2 + (2A+C)t + (2B+D)$
$\Rightarrow \quad A+C=2$, $B+D=-1$, $2A+C=3$, and $2B+D=-1 \quad \Rightarrow \quad A=1$, $C=1$, $B=0$, and $D=-1$.
Now

$$\int \frac{2t^3-t^2+3t-1}{(t^2+1)(t^2+2)}\,dt = \int \left(\frac{t}{t^2+1} + \frac{t-1}{t^2+2} \right) dt = \frac{1}{2}\int \frac{2t\,dt}{t^2+1} + \frac{1}{2}\int \frac{2t\,dt}{t^2+2} - \int \frac{dt}{t^2+2} = \tfrac{1}{2}\ln(t^2+1) + \tfrac{1}{2}\ln(t^2+2) - \tfrac{1}{\sqrt{2}}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}t\right) + C$$

or $\tfrac{1}{2}\ln\left((t^2+1)(t^2+2)\right) - \tfrac{\sqrt{2}}{2}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}t\right) + C$

38. $\dfrac{x^4}{x^4-1} = 1 + \dfrac{1}{x^4-1}$ and
$\dfrac{1}{x^4-1} = \dfrac{1}{(x-1)(x+1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1} \quad \Rightarrow$
$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$
Set $x=1$ to get $A = \frac{1}{4}$, and set $x=-1$ to get $B = -\frac{1}{4}$. Now take $x=0$ to get $1 = A-B-D = -D + \frac{1}{2}$, so that $D = -\frac{1}{2}$. Finally, equate the coefficients of x^3 to get $C=0$. Now

$$\int \frac{x^4\,dx}{x^4-1} = \int \left[1 + \frac{1/4}{x-1} - \frac{1/4}{x+1} - \frac{1/2}{x^2+1}\right] dx = x + \tfrac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \tfrac{1}{2}\tan^{-1}x + C$$

39. Let $u = x^4-x^2+1$. Then $du = (4x^3-2x)\,dx \quad \Rightarrow$

$$\int \frac{2x^3-x}{x^4-x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \tfrac{1}{2}\ln|x^4-x^2+1| + C = \tfrac{1}{2}\ln(x^4-x^2+1) + C$$

40. $\displaystyle\int_0^1 \frac{x-1}{x^2+2x+2}\,dx$

$$\begin{aligned}
&= \int_0^1 \frac{x+1}{x^2+2x+2}\,dx - \int_0^1 \frac{2}{x^2+2x+2}\,dx \\
&= \left[\tfrac{1}{2}\ln\left(x^2+2x+2\right)\right]_0^1 - 2\int_0^1 \frac{dx}{(x+1)^2+1} \\
&\qquad \left[\begin{array}{c}\text{set } u = x^2+2x+2,\ du = 2(x+1)\,dx \\ \text{in the first integral}\end{array}\right] \\
&= \tfrac{1}{2}(\ln 5 - \ln 2) - 2\left[\tan^{-1}(x+1)\right]_0^1 \\
&= \tfrac{1}{2}\ln\tfrac{5}{2} - 2\tan^{-1}2 + \tfrac{\pi}{2}
\end{aligned}$$

Or: Complete the square and let $u = x+1$.

41. Complete the square: $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$ and let $u = x + \frac{1}{2}$. Then

$$\begin{aligned}
\int_0^1 \frac{x}{x^2+x+1}\,dx &= \int_{1/2}^{3/2} \frac{u-1/2}{u^2+3/4}\,du \\
&= \int_{1/2}^{3/2} \frac{u}{u^2+3/4}\,du - \frac{1}{2}\int_{1/2}^{3/2} \frac{1}{u^2+3/4}\,du \\
&= \tfrac{1}{2}\ln\left(u^2+\tfrac{3}{4}\right) - \tfrac{1}{2}\tfrac{1}{\sqrt{3}/2}\left[\tan^{-1}\left(\tfrac{2}{\sqrt{3}}u\right)\right]_{1/2}^{3/2} \\
&= \tfrac{1}{2}\ln 3 - \tfrac{1}{\sqrt{3}}\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = \ln\sqrt{3} - \tfrac{\pi}{6\sqrt{3}}
\end{aligned}$$

42.
$$\begin{aligned}
\int_{-1/2}^{1/2} \frac{4x^2+5x+7}{4x^2+4x+5}\,dx &= \int_{-1/2}^{1/2}\left[1 + \frac{x+2}{4x^2+4x+5}\right]dx \\
&= [x]_{-1/2}^{1/2} + \int_{-1/2}^{1/2} \frac{x+1/2}{4x^2+4x+5}\,dx \\
&\qquad + \int_{-1/2}^{1/2} \frac{3/2}{4x^2+4x+5}\,dx \\
&= 1 + \left[\tfrac{1}{8}\ln\left(4x^2+4x+5\right)\right]_{-1/2}^{1/2} \\
&\qquad + \frac{3}{2}\int_{-1/2}^{1/2} \frac{dx}{(2x+1)^2+4} \\
&\qquad \left[\begin{array}{c}\text{set } u = 4x^2+4x+5,\ du = 8\left(x+\tfrac{1}{2}\right)dx \\ \text{in the first integral}\end{array}\right] \\
&= 1 + \tfrac{1}{8}(\ln 8 - \ln 4) + \frac{3}{2}\int_0^2 \frac{(1/2)\,du}{u^2+4} \\
&\qquad (\text{set } u = 2x+1, \text{ so } du = 2\,dx) \\
&= 1 + \tfrac{1}{8}\ln 2 + \tfrac{3}{4}\left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_0^2 \\
&= 1 + \tfrac{1}{8}\ln 2 + \tfrac{3\pi}{32}
\end{aligned}$$

43. $\displaystyle\frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} \quad\Rightarrow$

$$3x^3-x^2+6x-4 = (Ax+B)\left(x^2+2\right) + (Cx+D)\left(x^2+1\right)$$

Equating the coefficients gives $A + C = 3$, $B + D = -1$, $2A + C = 6$, and $2B + D = -4 \quad\Rightarrow$ $A = 3$, $C = 0$, $B = -3$, and $D = 2$. Now

$$\begin{aligned}
\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}\,dx &= 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2} \\
&= \tfrac{3}{2}\ln\left(x^2+1\right) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}x\right) + C
\end{aligned}$$

44. Let $u = \sin^2 x - 3\sin x + 2$. Then $du = (2\sin x\cos x - 3\cos x)\,dx$, so

$$\begin{aligned}
\int \frac{(2\sin x - 3)\cos x}{\sin^2 x - 3\sin x + 2}\,dx &= \int \frac{du}{u} = \ln|u| + C \\
&= \ln\left|\sin^2 x - 3\sin x + 2\right| + C
\end{aligned}$$

45. Let $u = \cos x$, then $du = -\sin x\,dx \quad\Rightarrow$

$$\begin{aligned}
\int \frac{\sin x\cos^2 x}{5+\cos^2 x}\,dx &= \int \frac{-u^2\,du}{5+u^2} = -\int\left[1 - \frac{5}{u^2+5}\right]du \\
&= -u + \tfrac{5}{\sqrt{5}}\tan^{-1}\left(\tfrac{1}{\sqrt{5}}u\right) + C \\
&= -\cos x + \sqrt{5}\tan^{-1}\left(\tfrac{1}{\sqrt{5}}\cos x\right) + C
\end{aligned}$$

46. $\displaystyle\frac{x^2+7x-6}{(x+1)(x^2-4x+7)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-4x+7} \quad\Rightarrow$ $x^2+7x-6 = A\left(x^2-4x+7\right) + (Bx+C)(x+1)$. Take $x = -1$ to get $-12 = 12A$ or $A = -1$. Then $2x^2+3x+1 = (Bx+C)(x+1)$, so $B = 2$ and $C = 1$. Now

$$\begin{aligned}
&\int \frac{x^2+7x-6}{(x+1)(x^2-4x+7)}\,dx \\
&= \int\left(\frac{-1}{x+1} + \frac{2x+1}{x^2-4x+7}\right)dx \\
&= -\ln|x+1| + \int \frac{2x-4}{x^2-4x+7}\,dx + 5\int \frac{dx}{(x-2)^2+3} \\
&= -\ln|x+1| + \ln\left(x^2-4x+7\right) \\
&\qquad + \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + C
\end{aligned}$$

47. $\displaystyle\frac{4x+1}{(x-3)(x^2+6x+12)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+6x+12} \quad\Rightarrow$ $4x+1 = A\left(x^2+6x+12\right) + (Bx+C)(x-3)$. Take $x = 3$ to get $13 = 39A \quad\Leftrightarrow\quad A = \frac{1}{3}$. Equate the terms of degree 2 and degree 0 to get $0 = \frac{1}{3} + B$ and $1 = 4 - 3C$, so $B = -\frac{1}{3}$ and $C = 1$. Now

$$\begin{aligned}
&\int \frac{(4x+1)\,dx}{(x-3)(x^2+6x+12)} \\
&= \int \frac{\frac{1}{3}\,dx}{x-3} - \frac{1}{3}\int \frac{x-3}{x^2+6x+12}\,dx \\
&= \tfrac{1}{3}\ln|x-3| - \frac{1}{3}\int \frac{x+3}{x^2+6x+12}\,dx \\
&\qquad + \frac{1}{3}\int \frac{6\,dx}{x^2+6x+12} \\
&= \tfrac{1}{3}\ln|x-3| - \tfrac{1}{6}\ln\left(x^2+6x+12\right) + 2\int \frac{dx}{(x+3)^2+3} \\
&= \tfrac{1}{3}\ln|x-3| - \tfrac{1}{6}\ln\left(x^2+6x+12\right) \\
&\qquad + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C
\end{aligned}$$

48. $\displaystyle\int \frac{x+1}{(x^2+x+2)^2}\,dx$

$$= \frac{1}{2}\int \frac{2x+1}{(x^2+x+2)^2}\,dx + \frac{1}{2}\int \frac{dx}{\left[\left(x+\frac{1}{2}\right)^2+\frac{7}{4}\right]^2}$$

$$= -\frac{1}{2(x^2+x+2)} + \frac{1}{2}\int \frac{du}{\left(u^2+\frac{7}{4}\right)^2} \quad \left(u = x+\tfrac{1}{2}\right)$$

$$= -\frac{1}{2(x^2+x+2)} + \frac{1}{2}\int \frac{\frac{\sqrt{7}}{2}\sec^2\theta\,d\theta}{\frac{49}{16}\sec^4\theta} \quad \left(u = \tfrac{\sqrt{7}}{2}\tan\theta\right)$$

$$= -\frac{1}{2(x^2+x+2)} + \tfrac{4\sqrt{7}}{49}\int \cos^2\theta\,d\theta$$

$$= -\frac{1}{2(x^2+x+2)} + \tfrac{2\sqrt{7}}{49}(\theta+\sin\theta\cos\theta) + C$$

$$= -\frac{1}{2(x^2+x+2)} + \frac{2\sqrt{7}}{49}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + \frac{2x+1}{14(x^2+x+2)} + C$$

$$= \frac{x-3}{7(x^2+x+2)} + \frac{2\sqrt{7}}{49}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C$$

49. $$\frac{3x^4-2x^3+20x^2-5x+34}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

Multiply by $x-1$ and set $x=1$ to get $A=2$. Then equate coefficients to get $B=1$, $C=-1$, $D=-1$, and $E=2$. So

$$\int \frac{3x^4-2x^3+20x^2-5x+34}{(x-1)(x^2+4)^2}\,dx$$

$$= 2\int \frac{dx}{x-1} + \int \frac{x\,dx}{x^2+4} - \int \frac{dx}{x^2+4} - \int \frac{x\,dx}{(x^2+4)^2} + 2\int \frac{dx}{(x^2+4)^2}$$

$$= 2\ln|x-1| + \tfrac{1}{2}\ln\left(x^2+4\right) - \tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}x\right) + \frac{1}{2(x^2+4)} + \frac{1}{8}\left[\tan^{-1}\left(\tfrac{1}{2}x\right) + \frac{2x}{x^2+4}\right] + C$$

where the last integral is evaluated by substituting $x = 2\tan\theta$.

50. Let $u = x^2+4$. Then

$$\int \frac{8x\,dx}{(x^2+4)^3} = 4\int \frac{2x\,dx}{(x^2+4)^3} = 4\int u^{-3}\,du$$

$$= -2u^{-2} + C = -\frac{2}{(x^2+4)^2} + C$$

51. Let $u = x^3+3x$. Then

$$\int \frac{(x^2+1)\,dx}{(x^3+3x)^2} = \frac{1}{3}\int u^{-2}\,du = -\frac{1}{3u} + C$$

$$= -\frac{1}{3(x^3+3x)} + C$$