6.6 IMPROPER INTEGRALS

A Click here for answers.

S Click here for solutions.

1–33 ▪ Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\displaystyle\int_2^{\infty} \frac{1}{(x+3)^{3/2}}\,dx$

2. $\displaystyle\int_0^{\infty} e^{-x}\,dx$

3. $\displaystyle\int_{-\infty}^{\infty} x^3\,dx$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt[3]{w-5}}\,dw$

5. $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x+3}}\,dx$

6. $\displaystyle\int_{-\infty}^{1} \frac{1}{(2x-3)^2}\,dx$

7. $\displaystyle\int_{\infty}^{-1} \frac{1}{\sqrt[3]{x-1}}\,dx$

8. $\displaystyle\int_{-\infty}^{\infty} x\,dx$

9. $\displaystyle\int_{-\infty}^{\infty} (2x^2 - x + 3)\,dx$

10. $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$

11. $\displaystyle\int_1^{\infty} \sin \pi x\,dx$

12. $\displaystyle\int_0^{\infty} \frac{5}{2x+3}\,dx$

13. $\displaystyle\int_{-\infty}^{3} \frac{1}{x^2+9}\,dx$

14. $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^2}\,dx$

15. $\displaystyle\int_0^{\infty} \frac{1}{(x+2)(x+3)}\,dx$

16. $\displaystyle\int_0^{\infty} \frac{x}{(x+2)(x+3)}\,dx$

17. $\displaystyle\int_0^{\infty} \cos x\,dx$

18. $\displaystyle\int_{-\infty}^{\pi/2} \sin 2\theta\,d\theta$

19. $\displaystyle\int_{-\infty}^{1} xe^{2x}\,dx$

20. $\displaystyle\int_0^{\infty} xe^{-x}\,dx$

21. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+6}$

22. $\displaystyle\int_0^{\infty} \frac{1}{2^x}\,dx$

23. $\displaystyle\int_0^{2} \frac{1}{4x-5}\,dx$

24. $\displaystyle\int_4^{5} \frac{1}{(5-x)^{2/5}}\,dx$

25. $\displaystyle\int_{\pi/4}^{\pi/2} \sec^2 x\,dx$

26. $\displaystyle\int_{\pi/4}^{\pi/2} \tan^2 x\,dx$

27. $\displaystyle\int_0^{\pi/4} \csc^2 t\,dt$

28. $\displaystyle\int_0^{\pi/4} \frac{\cos x}{\sqrt{\sin x}}\,dx$

29. $\displaystyle\int_{-2}^{2} \frac{1}{x^2-1}\,dx$

30. $\displaystyle\int_0^{2} \frac{x}{\sqrt{4-x^2}}\,dx$

31. $\displaystyle\int_0^{9} \frac{dx}{(x+9)\sqrt{x}}$

32. $\displaystyle\int_{\pi/4}^{3\pi/4} \tan x\,dx$

33. $\displaystyle\int_1^{e} \frac{1}{x\sqrt[4]{\ln x}}\,dx$

34–37 ▪ Sketch the region and find its area (if the area is finite).

34. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant (\ln x)/x^2\}$

35. $S = \left\{(x, y) \mid x \geqslant 0,\ 0 \leqslant y \leqslant 1/\sqrt{x+1}\right\}$

36. $S = \{(x, y) \mid 0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \tan x \sec x\}$

37. $S = \left\{(x, y) \mid 3 < x \leqslant 7,\ 0 \leqslant y \leqslant 1/\sqrt{x-3}\right\}$

38–40 ▪ Use the Comparison Theorem to determine whether the integral is convergent or divergent.

38. $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$

39. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x^3+1}}\,dx$

40. $\displaystyle\int_1^{\infty} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

6.5 ANSWERS

E Click here for exercises. S Click here for solutions.

1. $\frac{2}{\sqrt{5}}$

2. 1

3. Divergent

4. Divergent

5. Divergent

6. $\frac{1}{2}$

7. Divergent

8. Divergent

9. Divergent

10. $\frac{1}{3}$

11. Divergent

12. Divergent

13. $\frac{\pi}{4}$

14. 1

15. $-\ln\frac{2}{3}$

16. Divergent

17. Divergent

18. Divergent

19. $\frac{1}{4}e^2$

20. 1

21. $\dfrac{\pi}{\sqrt{2}}$

22. $\dfrac{1}{\ln 2}$

23. Divergent

24. $\frac{5}{3}$

25. Divergent

26. Divergent

27. Divergent

28. $2^{3/4}$

29. Divergent

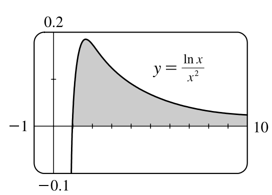
30. 2

31. $\frac{\pi}{6}$

32. Divergent

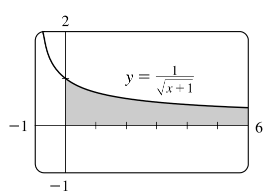
33. $\frac{4}{3}$

34.



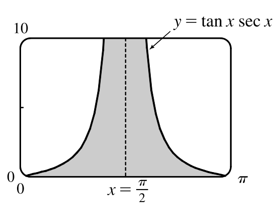
Area $= 1$

35.



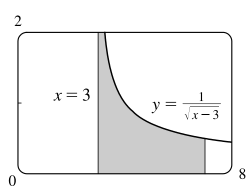
Area is infinite

36.



Area is infinite

37.



Area $= 4$

38. Convergent

39. Convergent

40. Divergent

6.6 SOLUTIONS

E Click here for exercises.

1. $\displaystyle\int_2^\infty \frac{dx}{(x+3)^{3/2}} = \lim_{t\to\infty}\int_2^t \frac{dx}{(x+3)^{3/2}}$

$\displaystyle= \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x+3}}\right]_2^t$

$\displaystyle= \lim_{t\to\infty}\left[\frac{-2}{\sqrt{t+3}}+\frac{2}{\sqrt{5}}\right] = \frac{2}{\sqrt{5}}$

2. $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$

$= \lim_{t\to\infty}\left(-e^{-t}+1\right) = 1$

3. $\int_{-\infty}^\infty x^3\,dx = \int_{-\infty}^0 x^3\,dx + \int_0^\infty x^3\,dx$, but

$\int_{-\infty}^0 x^3\,dx = \lim_{t\to-\infty}\left[\frac14 x^4\right]_t^0 = \lim_{t\to-\infty}\left(-\frac14 t^4\right)$

$= -\infty$

Divergent

4. $\displaystyle I = \int_{-\infty}^\infty \frac{1}{\sqrt[3]{w-5}}\,dw$

$\displaystyle= \lim_{a\to-\infty}\int_a^0 \frac{dw}{\sqrt[3]{w-5}} + \lim_{b\to5^-}\int_0^b \frac{dw}{\sqrt[3]{w-5}}$

$\displaystyle+ \lim_{c\to5^+}\int_c^{10} \frac{dw}{\sqrt[3]{w-5}} + \lim_{d\to\infty}\int_{10}^d \frac{dw}{\sqrt[3]{w-5}}$

(The values 0 and 10 could be any pair of values surrounding 5.) If any one of these four integrals diverges, then I diverges, and (for example)

$\displaystyle\lim_{d\to\infty}\int_{10}^d \frac{dw}{\sqrt[3]{w-5}} = \lim_{d\to\infty}\left[\frac32(w-5)^{2/3}\right]_{10}^d$

$\displaystyle= \lim_{d\to\infty}\left[\frac32(d-5)^{2/3}-\frac32(5)^{2/3}\right] = \infty$

Thus, I is divergent.

5. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x+3}} = \lim_{t\to\infty}\int_2^t \frac{dx}{\sqrt{x+3}} = \lim_{t\to\infty}\left[2\sqrt{x+3}\right]_2^t$

$= \lim_{t\to\infty}\left(2\sqrt{t+3}-2\sqrt{5}\right) = \infty$

Divergent

6. $\displaystyle\int_{-\infty}^1 \frac{dx}{(2x-3)^2} = \lim_{t\to-\infty}\frac12\int_t^1 \frac{2\,dx}{(2x-3)^2}$

$\displaystyle= \lim_{t\to-\infty}\frac12\left[-\frac{1}{2x-3}\right]_t^1$

$\displaystyle= \lim_{t\to-\infty}\left[\frac12+\frac{1}{2(2t-3)}\right] = \frac12$

7. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to-\infty}\int_t^{-1}(x-1)^{-1/3}\,dx$

$\displaystyle= \lim_{t\to-\infty}\left[\frac32(x-1)^{2/3}\right]_t^{-1}$

$\displaystyle= \lim_{t\to-\infty}\left[\frac32\sqrt[3]{4}-\frac32(t-1)^{2/3}\right] = -\infty$

Divergent

8. $\int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx$ and

$\int_{-\infty}^0 x\,dx = \lim_{t\to-\infty}\left[\frac12x^2\right]_t^0 = \lim_{t\to-\infty}\left(-\frac12t^2\right) = -\infty.$

Divergent

9. $\int_{-\infty}^\infty \left(2x^2-x+3\right)dx$

$= \int_{-\infty}^0 \left(2x^2-x+3\right)dx + \int_0^\infty \left(2x^2-x+3\right)dx$

but

$\int_{-\infty}^0 \left(2x^2-x+3\right)dx = \lim_{t\to-\infty}\left[\frac23x^3-\frac12x^2+3x\right]_t^0$

$= \lim_{t\to-\infty}\left[-\frac23t^3+\frac12t^2-3t\right] = \infty$

Divergent

10. $\int_{-\infty}^0 e^{3x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{3x}\,dx = \lim_{t\to-\infty}\left[\frac13e^{3x}\right]_t^0$

$= \lim_{t\to-\infty}\left[\frac13-\frac13e^{3t}\right] = \frac13$

11. $\int_1^\infty \sin\pi x\,dx = \lim_{t\to\infty} -\frac1\pi\left[\cos\pi x\right]_1^t$

$= -\frac1\pi\lim_{t\to\infty}\left(\cos\pi t+1\right)$

which does not exist. Divergent

12. $\displaystyle\int_0^\infty \frac{5\,dx}{2x+3} = \frac52\lim_{t\to\infty}\int_0^t \frac{2\,dx}{2x+3}$

$= \frac52\lim_{t\to\infty}\left[\ln(2x+3)\right]_0^t$

$= \frac52\lim_{t\to\infty}\left[\ln(2t+3)-\ln3\right] = \infty$

Divergent

13. $\displaystyle\int_{-\infty}^3 \frac{dx}{x^2+9} = \lim_{t\to-\infty}\left[\frac13\tan^{-1}\left(\frac13x\right)\right]_t^3$

$= \lim_{t\to-\infty}\frac13\left[\frac\pi4-\tan^{-1}\left(\frac13t\right)\right]$

$= \frac13\left(\frac\pi4+\frac\pi2\right) = \frac\pi4$

14. $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty}\int_e^t \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty}\left[-\frac{1}{\ln x}\right]_e^t$

$\displaystyle= \lim_{t\to\infty}\left[1-\frac{1}{\ln t}\right] = 1$

15. $\displaystyle\int_0^\infty \frac{dx}{(x+2)(x+3)} = \lim_{t\to\infty}\int_0^t\left[\frac{1}{x+2}-\frac{1}{x+3}\right]dx$

$\displaystyle= \lim_{t\to\infty}\left[\ln\left(\frac{x+2}{x+3}\right)\right]_0^t$

$\displaystyle= \lim_{t\to\infty}\left[\ln\left(\frac{t+2}{t+3}\right)-\ln\frac23\right]$

$= \ln1-\ln\frac23 = -\ln\frac23$

16. $\displaystyle\int_0^\infty \frac{x\,dx}{(x+2)(x+3)} = \lim_{t\to\infty}\int_0^t \left[\frac{-2}{x+2}+\frac{3}{x+3}\right]dx$
$\displaystyle = \lim_{t\to\infty}\left[3\ln(x+3)-2\ln(x+2)\right]_0^t$
$\displaystyle = \lim_{t\to\infty}\left[\ln\frac{(t+3)^3}{(t+2)^2}-\ln\tfrac{27}{4}\right]=\infty$

Divergent

17. $\int_0^\infty \cos x\,dx = \lim_{t\to\infty}[\sin x]_0^t = \lim_{t\to\infty}\sin t$, which does not exist. Divergent

18. $\int_{-\infty}^{\pi/2}\sin 2\theta\,d\theta = \lim_{t\to-\infty}\int_t^{\pi/2}\sin 2\theta\,d\theta$
$= \lim_{t\to-\infty}\left[-\tfrac{1}{2}\cos 2\theta\right]_t^{\pi/2}$
$= \lim_{t\to-\infty}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2t\right)$

This limit does not exist, so the integral is divergent.

19. $\int_{-\infty}^1 xe^{2x}\,dx = \lim_{t\to-\infty}\int_t^1 xe^{2x}\,dx$
$= \lim_{t\to-\infty}\left[\tfrac{1}{2}xe^{2x}-\tfrac{1}{4}e^{2x}\right]_t^1$ (by parts)
$= \lim_{t\to-\infty}\left[\tfrac{1}{2}e^2-\tfrac{1}{4}e^2-\tfrac{1}{2}te^{2t}+\tfrac{1}{4}e^{2t}\right]$
$= \tfrac{1}{4}e^2-0+0=\tfrac{1}{4}e^2$ since

$\displaystyle\lim_{t\to-\infty}te^{2t} = \lim_{t\to-\infty}\frac{t}{e^{-2t}} \overset{\text{H}}{=} \lim_{t\to-\infty}\frac{1}{-2e^{-2t}}$
$\displaystyle = \lim_{t\to-\infty}-\tfrac{1}{2}e^{2t}=0$

20. $\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\left[-xe^{-x}-e^{-x}\right]_0^t$
$= \lim_{t\to\infty}\left[1-(t+1)e^{-t}\right]$
$\displaystyle = 1-\lim_{t\to\infty}\frac{t+1}{e^t} \overset{\text{H}}{=} 1-\lim_{t\to\infty}\frac{1}{e^t}$
$= 1-0=1$

21. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+4x+6}$
$\displaystyle = \int_{-\infty}^0\frac{dx}{(x+2)^2+2}+\int_0^\infty\frac{dx}{(x+2)^2+2}$

Now
$\displaystyle\int_{-\infty}^0\frac{dx}{(x+2)^2+2} = \lim_{t\to-\infty}\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right)\right]_t^0$
$\displaystyle = \lim_{t\to-\infty}\frac{1}{\sqrt{2}}\left[\tan^{-1}\sqrt{2}-\tan^{-1}\left(\frac{t+2}{\sqrt{2}}\right)\right]$
$= \tfrac{1}{\sqrt{2}}\left(\tan^{-1}\sqrt{2}+\tfrac{\pi}{2}\right)$

and
$\displaystyle\int_0^\infty\frac{dx}{(x+2)^2+2} = \lim_{t\to\infty}\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right)\right]_0^t$
$= \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{2}-\tan^{-1}\sqrt{2}\right)$

Therefore, $\displaystyle\int_{-\infty}^\infty\frac{dx}{x^2+4x+6}=\frac{\pi}{\sqrt{2}}$.

22. $\displaystyle\int_0^\infty\frac{dx}{2^x} = \lim_{t\to\infty}\int_0^t 2^{-x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{\ln 2}2^{-x}\right]_0^t$
$\displaystyle = \frac{1}{\ln 2}\lim_{t\to\infty}\left(1-2^{-t}\right)=\frac{1}{\ln 2}$

23. $\displaystyle\int_0^2\frac{dx}{4x-5}=\int_0^{5/4}\frac{dx}{4x-5}+\int_{5/4}^2\frac{dx}{4x-5}$. But
$\displaystyle\int_0^{5/4}\frac{dx}{4x-5}=\lim_{t\to5/4^-}\left[\frac{1}{4}\ln|4x-5|\right]_0^t$
$\displaystyle = \lim_{t\to5/4^-}\tfrac{1}{4}\left[\ln|4t-5|-\ln 5\right]=-\infty$

Divergent

24. $\displaystyle\int_4^5\frac{dx}{(5-x)^{2/5}} = \lim_{t\to5^-}\left[-\tfrac{5}{3}(5-x)^{3/5}\right]_4^t$
$\displaystyle = \lim_{t\to5^-}\left[-\tfrac{5}{3}(5-t)^{3/5}+\tfrac{5}{3}\right]$
$= 0+\tfrac{5}{3}=\tfrac{5}{3}$

25. $\int_{\pi/4}^{\pi/2}\sec^2 x\,dx = \lim_{t\to\pi/2^-}[\tan x]_{\pi/4}^t$
$= \lim_{t\to\pi/2^-}(\tan t-1)=\infty$

Divergent

26. $\int_{\pi/4}^{\pi/2}\tan^2 x\,dx = \lim_{t\to\pi/2^-}\int_{\pi/4}^t\left(\sec^2 x-1\right)dx$
$= \lim_{t\to\pi/2^-}[\tan x-x]_{\pi/4}^t$
$= \tfrac{\pi}{4}-1+\lim_{t\to\pi/2^-}(\tan t-t)=\infty$

Divergent

27. $\int_0^{\pi/4}\csc^2 t\,dt = \lim_{s\to0^+}\int_s^{\pi/4}\csc^2 t\,dt$
$= \lim_{s\to0^+}[-\cot t]_s^{\pi/4}$
$= \lim_{s\to0^+}\left[-\cot\tfrac{\pi}{4}+\cot s\right]=\infty$

Divergent

28. $\displaystyle\int_0^{\pi/4}\frac{\cos x\,dx}{\sqrt{\sin x}} = \lim_{t\to0^+}\int_t^{\pi/4}\frac{\cos x\,dx}{\sqrt{\sin x}}$
$\displaystyle = \lim_{t\to0^+}\left[2\sqrt{\sin x}\right]_t^{\pi/4}$
$\displaystyle = \lim_{t\to0^+}\left(2\sqrt{\tfrac{1}{\sqrt{2}}}-2\sqrt{\sin t}\right)$
$\displaystyle = 2\sqrt{\tfrac{1}{\sqrt{2}}}=\tfrac{2}{2^{1/4}}=2^{3/4}$

29. $\displaystyle\int_{-2}^{2}\frac{dx}{x^2-1}=\int_{-2}^{-1}\frac{dx}{x^2-1}+\int_{-1}^{0}\frac{dx}{x^2-1}+\int_{0}^{1}\frac{dx}{x^2-1}+\int_{1}^{2}\frac{dx}{x^2-1}$

and

$$\int\frac{dx}{x^2-1}=\int\frac{dx}{(x-1)(x+1)}=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C, \text{ so}$$

$$\int_0^1\frac{dx}{x^2-1}=\lim_{t\to1^-}\left[\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_0^t=\lim_{t\to1^-}\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|=-\infty$$

Divergent

30. $\displaystyle\int_0^2\frac{x\,dx}{\sqrt{4-x^2}}=\lim_{t\to2^-}\int_0^t\frac{x\,dx}{\sqrt{4-x^2}}=\lim_{t\to2^-}\left[-\sqrt{4-x^2}\right]_0^t=\lim_{t\to2^-}\left(2-\sqrt{4-t^2}\right)=2$

31. $\displaystyle\int_0^9\frac{dx}{(x+9)\sqrt{x}}=\lim_{t\to0^+}\int_t^9\frac{dx}{(x+9)\sqrt{x}}$

$$=\lim_{t\to0^+}\int_{\sqrt{t}}^{3}\frac{2u\,du}{(u^2+9)\,u}\quad (u=\sqrt{x},\ x=u^2,\ dx=2u\,du)$$

$$=\lim_{t\to0^+}\int_{\sqrt{t}}^{3}\frac{2\,du}{u^2+9}=\lim_{t\to0^+}\left[\tfrac{2}{3}\tan^{-1}\left(\tfrac{1}{3}u\right)\right]_{\sqrt{t}}^{3}=\frac{2}{3}\left(\frac{\pi}{4}\right)=\frac{\pi}{6}$$

32. $\int_{\pi/4}^{3\pi/4}\tan x\,dx=\int_{\pi/4}^{\pi/2}\tan x\,dx+\int_{\pi/2}^{3\pi/4}\tan x\,dx$

But

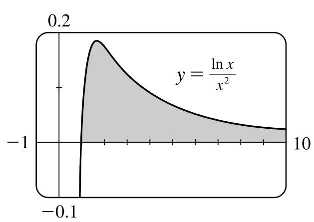
$$\int_{\pi/4}^{\pi/2}\tan x\,dx=\lim_{t\to(\pi/2)^-}\left[\ln|\sec x|\right]_{\pi/4}^{t}=\lim_{t\to(\pi/2)^-}\left[\ln(\sec t)-\ln\sqrt{2}\right]=\infty$$

Divergent

33. Let $u=\ln x$. Then $du=\dfrac{dx}{x}\ \Rightarrow$

$$\int_1^e\frac{dx}{x\sqrt[4]{\ln x}}=\lim_{t\to1^+}\int_t^e\frac{dx}{x\sqrt[4]{\ln x}}=\lim_{t\to1^+}\int_{\ln t}^{1}\frac{du}{\sqrt[4]{u}}=\lim_{t\to1^+}\left[\tfrac{4}{3}u^{3/4}\right]_{\ln t}^{1}=\lim_{t\to1^+}\left\{\tfrac{4}{3}\left[1-(\ln t)^{3/4}\right]\right\}=\tfrac{4}{3}$$

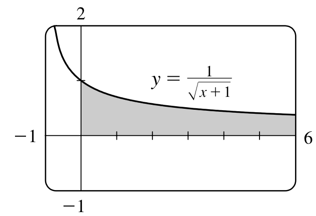
34.



We integrate by parts with $u=\ln x$, $dv=\left(1/x^2\right)dx$, $du=dx/x$, $v=-1/x$:

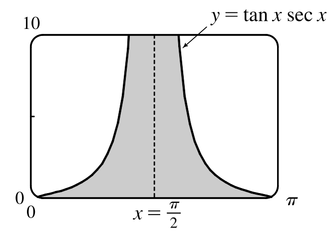
$$\int_1^\infty\frac{\ln x}{x^2}\,dx=\lim_{t\to\infty}\left(\left[-\frac{1}{x}\ln x\right]_1^t+\int_1^t\frac{1}{x^2}\,dx\right)=\lim_{t\to\infty}\left(-\frac{\ln t}{t}-\frac{1}{t}+1\right)=1$$

35.



Area$=\displaystyle\int_0^\infty\frac{dx}{\sqrt{x+1}}=\lim_{t\to\infty}\left[2\sqrt{x+1}\right]_0^t=\infty$, so the area is infinite.

36.



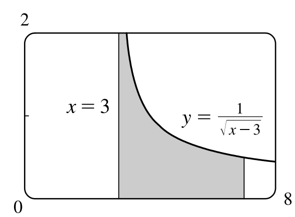
$$\int_0^\pi\tan x\sec x\,dx=\int_0^{\pi/2}\tan x\sec x\,dx+\int_{\pi/2}^{\pi}\tan x\sec x\,dx$$

But

$$\int_0^{\pi/2}\tan x\sec x\,dx=\lim_{t\to(\pi/2)^-}\int_0^t\tan x\sec x\,dx=\lim_{t\to(\pi/2)^-}\left[\sec x\right]_0^t=\lim_{t\to(\pi/2)^-}(\sec t-1)=\infty$$

Divergent

37.



$$\text{Area}=\int_3^7\frac{dx}{\sqrt{x-3}}=\lim_{t\to3^+}\left[2\sqrt{x-3}\right]_t^7=4-\lim_{t\to3^+}2\sqrt{t-3}=4-0=4$$

38. $\dfrac{\sin^2 x}{x^2}\le\dfrac{1}{x^2}$ on $[1,\infty)$. $\displaystyle\int_1^\infty\frac{dx}{x^2}$ is convergent by Equation 2, so $\displaystyle\int_1^\infty\frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

39. $\dfrac{1}{\sqrt{x^3+1}}\le\dfrac{1}{x^{3/2}}$ on $[1,\infty)$. $\displaystyle\int_1^\infty\frac{dx}{x^{3/2}}$ converges by Equation 2, so $\displaystyle\int_1^\infty\frac{dx}{\sqrt{x^3+1}}$ is convergent by the Comparison Theorem.

40. $\dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}>\dfrac{1}{\sqrt{x}}$ on $[1,\infty)$. $\displaystyle\int_1^\infty\frac{dx}{\sqrt{x}}$ is divergent by Equation 2 with $p=\frac{1}{2}\le1$, so $\displaystyle\int_1^\infty\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ is divergent by the Comparison Theorem.