7.3 VOLUMES BY CYLINDRICAL SHELLS

A Click here for answers.

S Click here for solutions.

1–7 ▪ Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis.

1. $y = x^2 - 6x + 10, \quad y = -x^2 + 6x - 6$

2. $y^2 = x, \quad x = 2y$

3. $y = x^2, \quad y = 4, \quad x = 0$

4. $y = x^2 - x^3, \quad y = 0$

5. $y = \sqrt{4 + x^2}, \quad y = 0, \quad x = 0, \quad x = 4$

6. $y = -x^2 + 4x - 3, \quad y = 0$

7. $y = x - 2, \quad y = \sqrt{x - 2}$

8–13 ▪ Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

8. $x = \sqrt[4]{y}, \quad x = 0, \quad y = 16$

9. $x = y^2, \quad x = 0, \quad y = 2, \quad y = 5$

10. $y = x, \quad x = 0, \quad x + y = 2$

11. $y = x^2, \quad y = 9$

12. $y^2 - 6y + x = 0, \quad x = 0$

13. $y = \sqrt{x}, \quad y = 0, \quad x + y = 2$

14–22 ▪ Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

14. $y = e^x,\; y = e^{-x},\; x = 1;$ about the y-axis

15. $y = e^x,\; x = 0,\; y = \pi;$ about the x-axis

16. $y = e^{-x},\; y = 0,\; x = -1,\; x = 0;$ about $x = 1$

17. $y = e^x,\; x = 0,\; y = 2;$ about $y = 1$

18. $y = \ln x,\; y = 0,\; x = e;$ about $y = 3$

19. $y = \sin x,\; y = 0,\; x = 2\pi,\; x = 3\pi;$ about the y-axis

20. $x = \cos y,\; x = 0,\; y = 0,\; y = \pi/4;$ about the x-axis

21. $y = -x^2 + 7x - 10,\; y = x - 2;$ about the x-axis

22. $x = 4 - y^2,\; x = 8 - 2y^2;$ about $y = 5$

23. The integral $\int_0^{\pi} 2\pi(4 - x)\sin^4 x\, dx$ represents the volume of a solid. Describe the solid.

24–25 ▪ Use a graph to estimate the x-coordinates of the points of intersection of the given curves. Then use this information to estimate the volume of the solid obtained by rotating about the y-axis the region enclosed by these curves.

24. $y = 0, \quad y = x + x^2 - x^4$

25. $y = x^4, \quad y = 3x - x^3$

26–27 ▪ The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.

26. $y = x^2 + x - 2,\; y = 0;$ about the x-axis

27. $y = x^2 - 3x + 2,\; y = 0;$ about the y-axis

7.3 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. 16π

2. $\frac{64}{15}\pi$

3. 8π

4. $\frac{1}{10}\pi$

5. $\frac{16}{3}\pi\left(5\sqrt{5}-1\right)$

6. $\frac{16\pi}{3}$

7. $\frac{4}{5}\pi$

8. $\frac{4096}{9}\pi$

9. $\frac{609}{2}\pi$

10. $\frac{2}{3}\pi$

11. $\frac{1944}{5}\pi$

12. 216π

13. $\frac{5}{6}\pi$

14. $V=\int_0^1 2\pi x\left(e^x-e^{-x}\right)dx$

15. $V=\int_1^\pi 2\pi y\cdot\ln y\,dy$

16. $V=\int_{-1}^0 2\pi\,(1-x)\,e^{-x}\,dx$

17. $V=2\pi\int_1^2 (y\ln y-\ln y)\,dy$

18. $V=2\pi\int_0^1 (3e-ey-3e^y+ye^y)\,dy$

19. $V=\int_{2\pi}^{3\pi} 2\pi x\sin x\,dx$

20. $V=\int_0^{\pi/4} 2\pi y\cos y\,dy$

21. $V=\pi\int_2^4 \left(x^4-14x^3+68x^2-136x+96\right)dx$

22. $V=\int_{-2}^2 2\pi\,(5-y)\left(4-y^2\right)dy$

23. Solid obtained by rotating the region under the curve $y=\sin^4 x$, above $y=0$, from $x=0$ to $x=\pi$, about the line $x=4$

24. 4.05

25. 4.62

26. $\frac{81}{10}\pi$

27. $\frac{1}{2}\pi$

7.3 SOLUTIONS

E Click here for exercises.

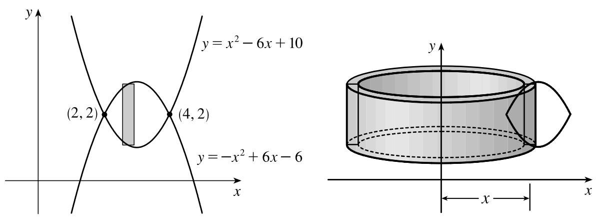
1. $V = \int_2^4 2\pi x\left[\left(-x^2+6x-6\right)-\left(x^2-6x+10\right)\right]dx$

$= 2\pi\int_2^4 x\left(-2x^2+12x-16\right)dx$

$= 4\pi\int_2^4\left(-x^3+6x^2-8x\right)dx$

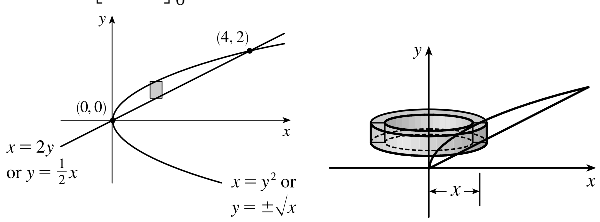
$= 4\pi\left[-\frac{1}{4}x^4+2x^3-4x^2\right]_2^4$

$= 4\pi\left[(-64+128-64)-(-4+16-16)\right] = 16\pi$



2. $V = \int_0^4 2\pi x\left(\sqrt{x}-\frac{1}{2}x\right)dx = 2\pi\int_0^4 x^{3/2}\,dx - \pi\int_0^4 x^2\,dx$

$= 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 - \pi\left[\frac{1}{3}x^3\right]_0^4 = \frac{4}{5}\pi(32) - \frac{64}{3}\pi = \frac{64}{15}\pi$



3. $V = \int_0^2 2\pi x\left(4-x^2\right)dx = 2\pi\int_0^2\left(4x-x^3\right)dx$

$= 2\pi\left[2x^2-\frac{1}{4}x^4\right]_0^2 = 2\pi(8-4) = 8\pi$

Note: If we integrated from -2 to 2, we would be generating the volume twice.

4. $V = \int_0^1 2\pi x\left(x^2-x^3\right)dx = 2\pi\int_0^1\left(x^3-x^4\right)dx$

$= 2\pi\left[\frac{1}{4}x^4-\frac{1}{5}x^5\right]_0^1 = 2\pi\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{1}{10}\pi$

5. $V = \int_0^4 2\pi x\sqrt{4+x^2}\,dx = \pi\int_0^4\sqrt{x^2+4}\,2x\,dx$

$= \left[\frac{2}{3}\pi\left(x^2+4\right)^{3/2}\right]_0^4 = \frac{2\pi}{3}\left(20\sqrt{20}-8\right)$

$= \frac{16}{3}\pi\left(5\sqrt{5}-1\right)$

6. $V = \int_1^3 2\pi x\left(-x^2+4x-3\right)dx$

$= 2\pi\left[-\frac{1}{4}x^4+\frac{4}{3}x^3-\frac{3}{2}x^2\right]_1^3$

$= 2\pi\left[\left(-\frac{81}{4}+36-\frac{27}{2}\right)-\left(-\frac{1}{4}+\frac{4}{3}-\frac{3}{2}\right)\right] = \frac{16\pi}{3}$

7. $V = \int_2^3 2\pi x\left[\sqrt{x-2}-(x-2)\right]dx$

$= \int_0^1 2\pi(u+2)\left(\sqrt{u}-u\right)du\ [u = x-2]$

$= 2\pi\int_0^1\left(u^{3/2}-u^2+2u^{1/2}-2u\right)du$

$= 2\pi\left[\frac{2}{5}u^{5/2}-\frac{1}{3}u^3+\frac{4}{3}u^{3/2}-u^2\right]_0^1$

$= 2\pi\left(\frac{2}{5}-\frac{1}{3}+\frac{4}{3}-1\right) = \frac{4}{5}\pi$

8. $V = \int_0^{16} 2\pi y\sqrt[4]{y}\,dy = 2\pi\int_0^{16} y^{5/4}\,dy$

$= 2\pi\left[\frac{4}{9}y^{9/4}\right]_0^{16} = \frac{8}{9}\pi(512-0) = \frac{4096}{9}\pi$

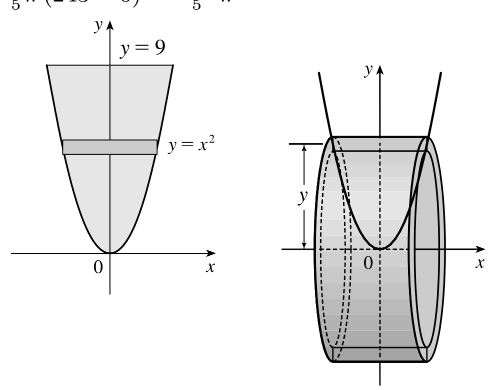
9. $V = \int_2^5 2\pi y\cdot y^2\,dy = 2\pi\left[\frac{1}{4}y^4\right]_2^5 = \frac{\pi}{2}(625-16) = \frac{609}{2}\pi$

10. $V = \int_0^1 2\pi y\left[(2-y)-y\right]dy = 4\pi\int_0^1 y(1-y)\,dy$

$= 4\pi\left[\frac{1}{2}y^2-\frac{1}{3}y^3\right]_0^1 = 4\pi\left(\frac{1}{6}\right) = \frac{2}{3}\pi$

11. $V = \int_0^9 2\pi y\cdot 2\sqrt{y}\,dy = 4\pi\int_0^9 y^{3/2}\,dy = 4\pi\left[\frac{2}{5}y^{5/2}\right]_0^9$

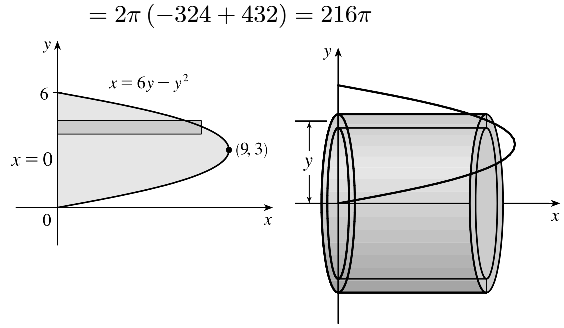
$= \frac{8}{5}\pi(243-0) = \frac{1944}{5}\pi$



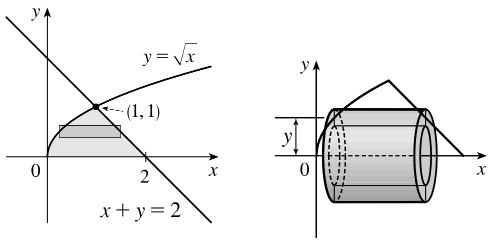
12. The two curves intersect at $(0,0)$ and $(0,6)$, so

$V = \int_0^6 2\pi y\left(-y^2+6y\right)dy = 2\pi\left[-\frac{1}{4}y^4+2y^3\right]_0^6$

$= 2\pi(-324+432) = 216\pi$



13. $V = \int_0^1 2\pi y\left[(2-y) - y^2\right] dy = 2\pi\left[y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4\right]_0^1$
$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{5}{6}\pi$



14. $V = \int_0^1 2\pi x\left(e^x - e^{-x}\right) dx$

15. $V = \int_1^\pi 2\pi y \cdot \ln y\, dy$

16. $V = \int_{-1}^0 2\pi\,(1-x)\,e^{-x}\, dx$

17. $V = \int_1^2 2\pi\,(y-1)\ln y\, dy = 2\pi\int_1^2 (y\ln y - \ln y)\, dy$

18. $V = \int_0^1 2\pi\,(3-y)\,(e - e^y)\, dy$
$= 2\pi\int_0^1 (3e - ey - 3e^y + ye^y)\, dy$

19. $V = \int_{2\pi}^{3\pi} 2\pi x \sin x\, dx$

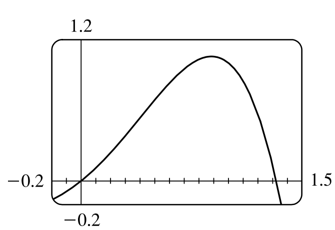
20. $V = \int_0^{\pi/4} 2\pi y \cos y\, dy$

21. $-x^2 + 7x - 10 = x - 2 \quad\Leftrightarrow\quad x^2 - 6x + 8 = 0 \quad\Leftrightarrow$
$x = 2$ or $4 \quad\Leftrightarrow\quad (x, y) = (2, 0)$ or $(4, 2)$. Use washers:
$V = \pi\int_2^4\left[\left(-x^2 + 7x - 10\right)^2 - (x-2)^2\right] dx$
$= \pi\int_2^4\left(x^4 - 14x^3 + 68x^2 - 136x + 96\right) dx$

22. $V = \int_{-2}^2 2\pi\,(5-y)\left[\left(8 - 2y^2\right) - \left(4 - y^2\right)\right] dy$
$= \int_{-2}^2 2\pi\,(5-y)\left(4 - y^2\right) dy$

23. The solid is obtained by rotating the region under the curve $y = \sin^4 x$, above $y = 0$, from $x = 0$ to $x = \pi$, about the line $x = 4$.

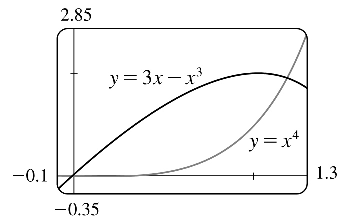
24.



From the graph, it appears that the curves intersect at $x = 0$ and at $x \approx 1.32$, with $x + x^2 - x^4 > 0$ on $(0, 1.32)$. So the volume of the solid obtained by rotating the region about the y-axis is

$$V \approx 2\pi\int_0^{1.32} x\left(x + x^2 - x^4\right) dx$$
$$= 2\pi\left[\tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^{1.32} \approx 4.05$$

25.



From the graph, it appears that the curves intersect at $x = 0$ and at $x \approx 1.17$, with $3x - x^3 > x^4$ on $(0, 1.17)$. So the volume of the solid obtained by rotation about the y-axis is

$$V \approx 2\pi\int_0^{1.17} x\left[\left(3x - x^3\right) - x^4\right] dx$$
$$= 2\pi\left[x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6\right]_0^{1.17} \approx 4.62$$

26. Use disks:
$$V = \int_{-2}^1 \pi\left(x^2 + x - 2\right)^2 dx$$
$$= \pi\int_{-2}^1\left(x^4 + 2x^3 - 3x^2 - 4x + 4\right) dx$$
$$= \pi\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 - x^3 - 2x^2 + 4x\right]_{-2}^1$$
$$= \pi\left[\left(\tfrac{1}{5} + \tfrac{1}{2} - 1 - 2 + 4\right) - \left(-\tfrac{32}{5} + 8 + 8 - 8 - 8\right)\right]$$
$$= \pi\left(\tfrac{33}{5} + \tfrac{3}{2}\right) = \tfrac{81}{10}\pi$$

27. Use shells:
$$V = \int_1^2 2\pi x\left(-x^2 + 3x - 2\right) dx$$
$$= 2\pi\int_1^2\left(-x^3 + 3x^2 - 2x\right) dx$$
$$= 2\pi\left[-\tfrac{1}{4}x^4 + x^3 - x^2\right]_1^2$$
$$= 2\pi\left[(-4 + 8 - 4) - \left(-\tfrac{1}{4} + 1 - 1\right)\right] = \tfrac{1}{2}\pi$$