7.4 ARC LENGTH

A Click here for answers.

S Click here for solutions.

1–4 ▪ Find the length of the arc of the given curve from point A to point B.

1. $y = 1 - x^{2/3}; \quad A(1, 0), \quad B(8, -3)$

2. $9y^2 = x(x - 3)^2; \quad A(0, 0), \quad B\left(4, \frac{2}{3}\right)$

3. $y^2 = (x - 1)^3; \quad A(1, 0), \; B(2, 1)$

4. $12xy = 4y^4 + 3; \quad A\left(\frac{7}{12}, 1\right), \; B\left(\frac{67}{24}, 2\right)$

5–9 ▪ Find the length of the curve.

5. $y = \frac{1}{3}(x^2 + 2)^{3/2}, \quad 0 \leqslant x \leqslant 1$

6. $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}, \quad 1 \leqslant x \leqslant 3$

7. $y = \ln(\sin x), \quad \pi/6 \leqslant x \leqslant \pi/3$

8. $y = \ln(1 - x^2), \quad 0 \leqslant x \leqslant \frac{1}{2}$

9. $y = \ln(\cos x), \quad 0 \leqslant x \leqslant \pi/4$

10–12 ▪ Set up, but do not evaluate, an integral for the length of the curve.

10. $y = \tan x, \quad 0 \leqslant x \leqslant \pi/4$

11. $y = x^3, \quad 0 \leqslant x \leqslant 1$

12. $y = e^x \cos x, \quad 0 \leqslant x \leqslant \pi/2$

13–16 ▪ Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve.

13. $y = x^3, \quad 0 \leqslant x \leqslant 1$

14. $y = 1/x, \quad 1 \leqslant x \leqslant 2$

15. $y = \sin x, \quad 0 \leqslant x \leqslant \pi$

16. $y = \tan x, \quad 0 \leqslant x \leqslant \pi/4$

7.4 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{1}{27}\left(80\sqrt{10}-13\sqrt{13}\right)$
2. $\frac{14}{3}$
3. $\frac{13\sqrt{13}-8}{27}$
4. $\frac{59}{24}$
5. $\frac{4}{3}$
6. $\frac{181}{9}$
7. $\ln\left(1+\frac{2}{\sqrt{3}}\right)$
8. $\ln 3-\frac{1}{2}$
9. $\ln\left(\sqrt{2}+1\right)$
10. $\int_0^{\pi/4}\sqrt{1+\sec^4 x}\,dx$
11. $\int_0^1\sqrt{1+9x^4}\,dx$
12. $L=\int_0^{\pi/2}\sqrt{1+e^{2x}\left(1-\sin 2x\right)}\,dx$
13. 1.548
14. 1.132
15. 3.820
16. 1.278

7.4 SOLUTIONS

E Click here for exercises.

1. $y = 1 - x^{2/3} \Rightarrow dy/dx = -\frac{2}{3}x^{-1/3} \Rightarrow$
$1 + (dy/dx)^2 = 1 + \frac{4}{9}x^{-2/3}$. So
$$L = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\,dx = \int_1^8 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx$$
$$= \tfrac{1}{18}\int_{13}^{40}\sqrt{u}\,du \quad (u = 9x^{2/3}+4,\, du = 6x^{-1/3}\,dx)$$
$$= \tfrac{1}{27}\left[u^{3/2}\right]_{13}^{40} = \tfrac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$
Or: Let $u = x^{1/3}$.

2. $9y^2 = x(x-3)^2$, $3y = x^{1/2}(x-3)$, $y = \frac{1}{3}x^{3/2} - x^{1/2}$
$\Rightarrow y' = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2} \Rightarrow (y')^2 = \frac{1}{4}x - \frac{1}{2} + \frac{1}{2}x^{-1}$
$\Rightarrow 1 + (y')^2 = \frac{1}{4}x + \frac{1}{2} + \frac{1}{4}x^{-1} \Rightarrow$
$\sqrt{1+(y')^2} = \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}$. So
$$L = \int_0^4 \left(\tfrac{1}{2}x^{1/2} + \tfrac{1}{2}x^{-1/2}\right)dx = \tfrac{1}{2}\left[\tfrac{2}{3}x^{3/2} + 2x^{1/2}\right]_0^4$$
$$= \tfrac{1}{2}\left(\tfrac{16}{3} + 4\right) = \tfrac{14}{3}$$

3. $y^2 = (x-1)^3$, $y = (x-1)^{3/2}$
$\Rightarrow dy/dx = \frac{3}{2}(x-1)^{1/2} \Rightarrow$
$1 + (dy/dx)^2 = 1 + \frac{9}{4}(x-1)$. So
$$L = \int_1^2 \sqrt{1 + \tfrac{9}{4}(x-1)}\,dx = \int_1^2 \sqrt{\tfrac{9}{4}x - \tfrac{5}{4}}\,dx$$
$$= \left[\tfrac{4}{9}\cdot\tfrac{2}{3}\left(\tfrac{9}{4}x - \tfrac{5}{4}\right)^{3/2}\right]_1^2 = \tfrac{13\sqrt{13}-8}{27}$$

4. $12xy = 4y^4 + 3$, $x = \dfrac{y^3}{3} + \dfrac{y^{-1}}{4} \Rightarrow \dfrac{dx}{dy} = y^2 - \dfrac{y^{-2}}{4}$,
so $\left(\dfrac{dx}{dy}\right)^2 = y^4 - \dfrac{1}{2} + \dfrac{y^{-4}}{16} \Rightarrow$
$1 + \left(\dfrac{dx}{dy}\right)^2 = y^4 + \dfrac{1}{2} + \dfrac{y^{-4}}{16} \Rightarrow$
$\sqrt{1 + \left(\dfrac{dx}{dy}\right)^2} = y^2 + \dfrac{y^{-2}}{4}$. So
$$L = \int_1^2 \left(y^2 + \frac{y^{-2}}{4}\right)dy = \left[\frac{y^3}{3} - \frac{1}{4y}\right]_1^2$$
$$= \left(\tfrac{8}{3} - \tfrac{1}{8}\right) - \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = \tfrac{59}{24}$$

5. $y = \frac{1}{3}\left(x^2+2\right)^{3/2} \Rightarrow$
$dy/dx = \frac{1}{2}\left(x^2+2\right)^{1/2}(2x) = x\sqrt{x^2+2} \Rightarrow$
$1 + (dy/dx)^2 = 1 + x^2\left(x^2+2\right) = \left(x^2+1\right)^2$. So
$L = \int_0^1 \left(x^2+1\right)dx = \left[\frac{1}{3}x^3 + x\right]_0^1 = \frac{4}{3}$.

6. $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2} \Rightarrow \dfrac{dy}{dx} = x^3 - \dfrac{1}{4x^3} \Rightarrow$
$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + x^6 - \dfrac{1}{2} + \dfrac{1}{16x^6} = x^6 + \dfrac{1}{2} + \dfrac{1}{16x^6}$. So
$$L = \int_1^3 \left(x^3 + \tfrac{1}{4}x^{-3}\right)dx = \left[\tfrac{1}{4}x^4 - \tfrac{1}{8}x^{-2}\right]_1^3$$
$$= \left(\tfrac{81}{4} - \tfrac{1}{72}\right) - \left(\tfrac{1}{4} - \tfrac{1}{8}\right) = \tfrac{181}{9}$$

7. $y = \ln(\sin x) \Rightarrow \dfrac{dy}{dx} = \dfrac{\cos x}{\sin x} = \cot x \Rightarrow$
$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \cot^2 x = \csc^2 x$. So
$$L = \int_{\pi/6}^{\pi/3} \csc x\,dx = \left[\ln(\csc x - \cot x)\right]_{\pi/6}^{\pi/3}$$
$$= \ln\left(\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right) - \ln\left(2 - \sqrt{3}\right)$$
$$= \ln\frac{1}{\sqrt{3}\left(2-\sqrt{3}\right)} = \ln\frac{2+\sqrt{3}}{\sqrt{3}} = \ln\left(1 + \tfrac{2}{\sqrt{3}}\right)$$

8. $y = \ln\left(1-x^2\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{-2x}{1-x^2} \Rightarrow$
$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{4x^2}{\left(1-x^2\right)^2} = \dfrac{\left(1+x^2\right)^2}{\left(1-x^2\right)^2}$. So
$$L = \int_0^{1/2} \frac{1+x^2}{1-x^2}\,dx = \int_0^{1/2}\left[-1 + \frac{2}{(1-x)(1+x)}\right]dx$$
$$= \int_0^{1/2}\left[-1 + \frac{1}{1+x} + \frac{1}{1-x}\right]dx$$
$$= \left[-x + \ln(1+x) - \ln(1-x)\right]_0^{1/2}$$
$$= -\tfrac{1}{2} + \ln\tfrac{3}{2} - \ln\tfrac{1}{2} - 0 = \ln 3 - \tfrac{1}{2}$$

9. $y = \ln(\cos x) \Rightarrow y' = \dfrac{1}{\cos x}(-\sin x) = -\tan x$
$\Rightarrow 1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$. So
$L = \int_0^{\pi/4} \sec x\,dx = \ln(\sec x + \tan x)\big|_0^{\pi/4} = \ln\left(\sqrt{2}+1\right)$

10. $y = \tan x \Rightarrow 1 + (y')^2 = 1 + \sec^4 x$. So
$L = \int_0^{\pi/4} \sqrt{1 + \sec^4 x}\,dx$.

11. $y = x^3 \Rightarrow y' = 3x^2 \Rightarrow 1 + (y')^2 = 1 + 9x^4$. So
$L = \int_0^1 \sqrt{1 + 9x^4}\,dx$.

12. $y = e^x\cos x \Rightarrow y' = e^x(\cos x - \sin x) \Rightarrow$
$$1 + (y')^2 = 1 + e^{2x}\left(\cos^2 x - 2\cos x\sin x + \sin^2 x\right)$$
$$= 1 + e^{2x}(1 - \sin 2x)$$
So $L = \int_0^{\pi/2} \sqrt{1 + e^{2x}(1 - \sin 2x)}\,dx$.

13. $y = x^3 \quad \Rightarrow \quad 1 + (y')^2 = 1 + \left(3x^2\right)^2 = 1 + 9x^4 \quad \Rightarrow$
$L = \int_0^1 \sqrt{1 + 9x^4}\, dx$. Let $f(x) = \sqrt{1 + 9x^4}$. Then by Simpson's Rule with $n = 10$,

$$L \approx \frac{1/10}{3}\left[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + \cdots + 2f(0.8) + 4f(0.9) + f(1)\right] \approx 1.548$$

14. $y = \dfrac{1}{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{x^2} \quad \Rightarrow \quad 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{1}{x^4}$.

Therefore, with $f(x) = \sqrt{1 + \dfrac{1}{x^4}}$,

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + 1/x^4}\, dx \approx S_{10} \\ &= \tfrac{2-1}{10 \cdot 3}\left[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2)\right] \\ &\approx 1.132104 \end{aligned}$$

15. $y = \sin x$, $1 + (dy/dx)^2 = 1 + \cos^2 x$,
$L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx$. Let $g(x) = \sqrt{1 + \cos^2 x}$. Then

$$\begin{aligned} L &\approx \tfrac{\pi/10}{3}\left[g(0) + 4g\left(\tfrac{\pi}{10}\right) + 2g\left(\tfrac{\pi}{5}\right) + 4g\left(\tfrac{3\pi}{10}\right) + 2g\left(\tfrac{2\pi}{5}\right) + 4g\left(\tfrac{\pi}{2}\right) + 2g\left(\tfrac{3\pi}{5}\right) + 4g\left(\tfrac{7\pi}{10}\right) + 2g\left(\tfrac{4\pi}{5}\right) + 4g\left(\tfrac{9\pi}{10}\right) + g(\pi)\right] \\ &\approx 3.820 \end{aligned}$$

16. $y = \tan x \quad \Rightarrow \quad 1 + (y')^2 = 1 + \sec^4 x$. So
$L = \int_0^{\pi/4} \sqrt{1 + \sec^4 x}\, dx$. Let $g(x) = \sqrt{1 + \sec^4 x}$. Then

$$\begin{aligned} L &\approx \tfrac{\pi/40}{3}\left[g(0) + 4g\left(\tfrac{\pi}{40}\right) + 2g\left(\tfrac{2\pi}{40}\right) + 4g\left(\tfrac{3\pi}{40}\right) + 2g\left(\tfrac{4\pi}{40}\right) + 4g\left(\tfrac{5\pi}{40}\right) + 2g\left(\tfrac{6\pi}{40}\right) + 4g\left(\tfrac{7\pi}{40}\right) + 2g\left(\tfrac{8\pi}{40}\right) + 4g\left(\tfrac{9\pi}{40}\right) + g\left(\tfrac{\pi}{4}\right)\right] \\ &\approx 1.278 \end{aligned}$$