7.6 APPLICATIONS TO PHYSICS AND ENGINEERING

A Click here for answers.

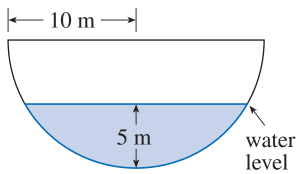
S Click here for solutions.

1. A particle is moved along the x-axis by a force that measures $5x^2 + 1$ pounds at a point x feet from the origin. Find the work done in moving the particle from the origin to a distance of 10 ft.

2. A uniform cable hanging over the edge of a tall building is 40 ft long and weighs 60 lb. How much work is required to pull 10 ft of the cable to the top?

3–5 ■ The end of a tank containing water is vertical and has the indicated shape. Explain how to approximate the hydrostatic force against the end of the tank by a Riemann sum. Then express the force as an integral and evaluate it.

3.



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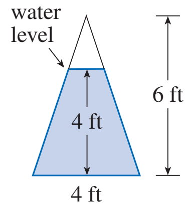
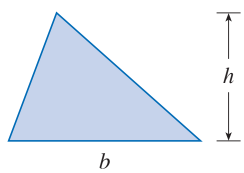


5.



6. The masses $m_1 = 4$ and $m_2 = 8$ are located at the points $P_1(-1, 2)$ and $P_2(2, 4)$. Find the moments M_x and M_y and the center of mass of the system.

7–8 ■ Sketch the region bounded by the curves, and visually estimate the location of the centroid. Then find the exact coordinates of the centroid.

7. $y = x^2, \quad y = 0, \quad x = 2$

8. $y = \sqrt{x}, \quad y = 0, \quad x = 9$

9–10 ■ Find the centroid of the region bounded by the curves.

9. $y = \sin 2x, \quad y = 0, \quad x = 0, \quad x = \pi/2$

10. $y = \ln x, \quad y = 0, \quad x = e$

7.6 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{5030}{3}$ ft-lb
2. 525 ft-lb
3. 1.23×10^6 N
4. 1.56×10^3 lb
5. $\frac{1}{3}\rho g b h^2$
6. 40, 12, $\left(1, \frac{10}{3}\right)$
7. $(1.5, 1.2)$
8. $\left(\frac{27}{5}, \frac{9}{8}\right)$
9. $\left(\frac{\pi}{4}, \frac{\pi}{8}\right)$
10. $\left(\dfrac{e^2+1}{4}, \dfrac{e-2}{2}\right)$

7.6 SOLUTIONS

E Click here for exercises.

1. By Equation 4,

$$\begin{aligned} W &= \int_a^b f(x)\,dx \\ &= \int_0^{10} \left(5x^2+1\right) dx \\ &= \left[\tfrac{5}{3}x^3 + x\right]_0^{10} = \tfrac{5000}{3} + 10 \\ &= \tfrac{5030}{3} \text{ ft-lb} \end{aligned}$$

2. Each part of the top 10 ft of cable is lifted a distance x_i^* equal to its distance from the top. The cable weighs $\frac{60}{40} = 1.5$ lb/ft, so the work done on the ith subinterval is $\frac{3}{2}x_i^*\,\Delta x$. The remaining 30 ft of cable is lifted 10 ft. Thus,

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{3}{2}x_i^*\,\Delta x + \tfrac{3}{2}\cdot 10\,\Delta x\right) \\ &= \int_0^{10} \tfrac{3}{2}x\,dx + \int_{10}^{40} \tfrac{3}{2}\cdot 10\,dx \\ &= \left[\tfrac{3}{4}x^2\right]_0^{10} + [15x]_{10}^{40} \\ &= \tfrac{3}{4}(100) + 15(30) \\ &= 75 + 450 \\ &= 525 \text{ ft-lb} \end{aligned}$$

3. In the middle of the figure, draw a vertical x-axis that increases in the downward direction. The area of the ith rectangular strip is $2\sqrt{100-(x_i^*)^2}\,\Delta x$ and the pressure on the strip is $\rho g\,(x_i^*-5)$ $[\rho = 1000 \text{ kg/m}^3$ and $g = 9.8 \text{ m/s}^2]$. Thus, the hydrostatic force on the ith strip is the product $\rho g\,(x_i^*-5)\,2\sqrt{100-(x_i^*)^2}\,\Delta x$.

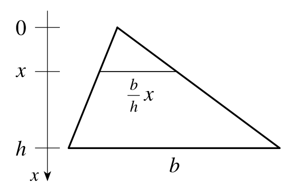
$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho g\,(x_i^*-5)\,2\sqrt{100-(x_i^*)^2}\,\Delta x \\ &= \int_5^{10} \rho g\,(x-5)\cdot 2\sqrt{100-x^2}\,dx \\ &= \rho g \int_5^{10} 2x\sqrt{100-x^2}\,dx - 10\rho g \int_5^{10} \sqrt{100-x^2}\,dx \\ &\overset{30}{=} -\rho g\left[\tfrac{2}{3}\left(100-x^2\right)^{3/2}\right]_5^{10} \\ &\qquad - 10\rho g\left[\tfrac{1}{2}x\sqrt{100-x^2} + 50\sin^{-1}(x/10)\right]_5^{10} \\ &= \tfrac{2}{3}\rho g\,(75)^{3/2} - 10\rho g\left[50\left(\tfrac{\pi}{2}\right) - \tfrac{5}{2}\sqrt{75} - 50\left(\tfrac{\pi}{6}\right)\right] \\ &= 250\rho g\left(\tfrac{3\sqrt{3}}{2} - \tfrac{2\pi}{3}\right) \approx 1.23\times 10^6 \text{ N} \end{aligned}$$

4. Place an x-axis as in Problem 3. Using similar triangles, $\dfrac{4 \text{ ft wide}}{6 \text{ ft high}} = \dfrac{w \text{ ft wide}}{x_i^* \text{ ft high}}$, so $w = \frac{4}{6}x_i^*$ and the area of the ith rectangular strip is $\frac{4}{6}x_i^*\,\Delta x$. The pressure on the strip is $\delta\,(x_i^*-2)$ $[\delta = \rho g = 62.5 \text{ lb/ft}^3]$ and the hydrostatic force is $\delta\,(x_i^*-2)\,\frac{4}{6}x_i^*\,\Delta x$.

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} \delta\,(x_i^*-2)\,\tfrac{4}{6}x_i^*\,\Delta x = \int_2^6 \delta\,(x-2)\,\tfrac{2}{3}x\,dx \\ &= \tfrac{2}{3}\delta \int_2^6 \left(x^2-2x\right) dx = \tfrac{2}{3}\delta\left[\tfrac{1}{3}x^3 - x^2\right]_2^6 \\ &= \tfrac{2}{3}\delta\left[\,36 - \left(-\tfrac{4}{3}\right)\right] = \tfrac{224}{9}\delta \approx 1.56\times 10^3 \text{ lb} \end{aligned}$$

5. This is like Problem 4, except that the area of the ith rectangular strip is $\dfrac{b}{h}x_i^*\,\Delta x$ and the pressure on the strip is δx_i^*.

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \frac{b}{h} x_i^*\,\Delta x = \int_0^h \rho g x \cdot \frac{b}{h}x\,dx \\ &= \left[\frac{\rho g b}{3h}x^3\right]_0^h = \tfrac{1}{3}\rho g b h^2 \end{aligned}$$



6. $m_1 = 4$, $m_2 = 8$; $P_1(-1,2)$, $P_2(2,4)$.

$m = \sum_{i=1}^{2} m_i = m_1 + m_2 = 12.$

$M_x = \sum_{i=1}^{2} m_i y_i = 4\cdot 2 + 8\cdot 4 = 40;$

$M_y = \sum_{i=1}^{2} m_i x_i = 4\cdot(-1) + 8\cdot 2 = 12;$

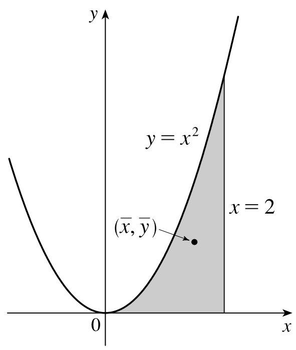
$\overline{x} = M_y/m = 1$ and $\overline{y} = M_x/m = \frac{10}{3}$, so the center of mass of the system is $(\overline{x},\overline{y}) = \left(1, \frac{10}{3}\right)$.

7. $A = \int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3}$,

$\overline{x} = A^{-1}\int_0^2 x \cdot x^2\,dx = \frac{3}{8}\left[\frac{1}{4}x^4\right]_0^2 = \frac{3}{8}\cdot 4 = \frac{3}{2}$,

$\overline{y} = A^{-1}\int_0^2 \frac{1}{2}\left(x^2\right)^2\,dx = \frac{3}{8}\cdot\frac{1}{2}\left[\frac{1}{5}x^5\right]_0^2 = \frac{3}{16}\cdot\frac{32}{5} = \frac{6}{5}$

Centroid $(\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{6}{5}\right) = (1.5, 1.2)$

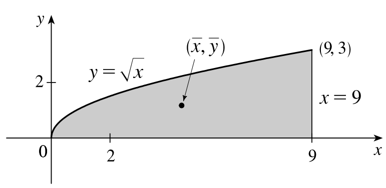


8. $A = \int_0^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^9 = \frac{2}{3}\cdot 27 = 18$,

$\overline{x} = \frac{1}{A}\int_0^9 x\sqrt{x}\,dx = \frac{1}{18}\left[\frac{2}{5}x^{5/2}\right]_0^9 = \frac{1}{18}\cdot\frac{2}{5}\cdot 243 = \frac{27}{5}$,

$\overline{y} = \frac{1}{A}\int_0^9 \frac{1}{2}\left(\sqrt{x}\right)^2\,dx = \frac{1}{18}\cdot\frac{1}{2}\left[\frac{1}{2}x^2\right]_0^9 = \frac{81}{72} = \frac{9}{8}$.

Centroid $(\overline{x}, \overline{y}) = \left(\frac{27}{5}, \frac{9}{8}\right) = (5.4, 1.125)$



9. From the figure we see that $\overline{x} = \frac{\pi}{4}$ (halfway from $x = 0$ to $\frac{\pi}{2}$). Now

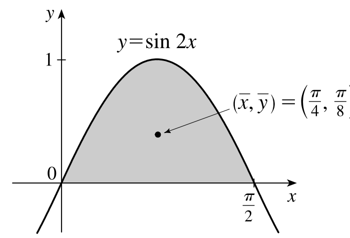
$A = \int_0^{\pi/2} \sin 2x\,dx = -\frac{1}{2}\left[\cos 2x\right]_0^{\pi/2} = -\frac{1}{2}(-1-1) = 1$

so

$\overline{y} = \frac{1}{A}\int_0^{\pi/2} \frac{1}{2}\sin^2 2x\,dx = \frac{1}{1}\int_0^{\pi/2} \frac{1}{2}\cdot\frac{1}{2}(1-\cos 4x)\,dx$

$= \frac{1}{4}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{\pi}{8}$.

Centroid $(\overline{x}, \overline{y}) = \left(\frac{\pi}{4}, \frac{\pi}{8}\right)$.



10. $A = \int_1^e \ln x\,dx = \left[x\ln x - x\right]_1^e = 0 - (-1) = 1$,

$$\overline{x} = \frac{1}{A}\int_1^e x\ln x\,dx = \left[\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_1^e$$

$$= \left(\tfrac{1}{2}e^2 - \tfrac{1}{4}e^2\right) - \left(-\tfrac{1}{4}\right) = \frac{e^2+1}{4}$$

$\overline{y} = \dfrac{1}{A}\displaystyle\int_1^e \frac{(\ln x)^2}{2}\,dx = \frac{1}{2}\int_1^e (\ln x)^2\,dx$. To evaluate $\int (\ln x)^2\,dx$, take $u = \ln x$ and $dv = \ln x\,dx$, so that $du = 1/x\,dx$ and $v = x\ln x - x$. Then

$$\begin{aligned}
\int (\ln x)^2\,dx &= x(\ln x)^2 - x(\ln x) - \int (x\ln x - x)\,\tfrac{1}{x}\,dx \\
&= x(\ln x)^2 - x(\ln x) - \int (\ln x - 1)\,dx \\
&= x(\ln x)^2 - x\ln x - x\ln x + x + x + C \\
&= x(\ln x)^2 - 2x\ln x + 2x + C
\end{aligned}$$

Thus

$$\begin{aligned}
\overline{y} &= \tfrac{1}{2}\left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^e \\
&= \tfrac{1}{2}\left[(e - 2e + 2e) - (0 - 0 + 2)\right] = \frac{e-2}{2}
\end{aligned}$$

So $(\overline{x}, \overline{y}) = \left(\dfrac{e^2+1}{4}, \dfrac{e-2}{2}\right)$.