8.4 OTHER CONVERGENCE TESTS

A Click here for answers.

S Click here for solutions.

1–14 ▪ Test the series for convergence or divergence.

1. $\frac{3}{5} - \frac{3}{6} + \frac{3}{7} - \frac{3}{8} + \frac{3}{9} - \cdots$

2. $-5 - \frac{5}{2} + \frac{5}{5} - \frac{5}{8} + \frac{5}{11} - \frac{5}{14} + \cdots$

3. $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots$

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{5n+1}$

7. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \ln n}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

9. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+1}$

10. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n+1}$

11. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n^2}{4n^2+1}$

12. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$

13. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

14. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{\ln n}}$

15–18 ▪ Approximate the sum of the series to the indicated accuracy.

15. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$ (four decimal places)

16. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ (four decimal places)

17. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$ (four decimal places)

18. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6}$ (five decimal places)

19–38 ▪ Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$

20. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$

21. $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$

25. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{3n-4}$

26. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2+1}$

27. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n 3^{n+1}}$

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^{n-1}}{(n+1)^2 4^{n+2}}$

29. $\sum_{n=1}^{\infty} \frac{(n+1)5^n}{n 3^{2n}}$

30. $\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$

31. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^3}$

32. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/6)}{n\sqrt{n}}$

33. $\sum_{n=1}^{\infty} \frac{n!}{(-10)^n}$

34. $\sum_{n=1}^{\infty} \frac{8-n^3}{n!}$

35. $\sum_{n=1}^{\infty} \frac{(-n)^n}{5^{2n+3}}$

36. $\sum_{n=1}^{\infty} \frac{(-2)^n n^2}{(n+2)!}$

37. $\sum_{n=1}^{\infty} \frac{(n+2)!}{n!\,10^n}$

38. $\frac{1}{3} + \frac{1 \cdot 4}{3 \cdot 5} + \frac{1 \cdot 4 \cdot 7}{3 \cdot 5 \cdot 7} + \frac{1 \cdot 4 \cdot 7 \cdot 10}{3 \cdot 5 \cdot 7 \cdot 9} + \cdots + \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} + \cdots$

8.4 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. Convergent

2. Convergent

3. Divergent

4. Convergent

5. Convergent

6. Divergent

7. Convergent

8. Convergent

9. Divergent

10. Divergent

11. Divergent

12. Convergent

13. Convergent

14. Convergent

15. 0.8415

16. 0.5403

17. 0.6065

18. 0.98555

19. Absolutely convergent

20. Divergent

21. Absolutely convergent

22. Absolutely convergent

23. Absolutely convergent

24. Absolutely convergent

25. Divergent

26. Divergent

27. Absolutely convergent

28. Divergent

29. Absolutely convergent

30. Absolutely convergent

31. Absolutely convergent

32. Absolutely convergent

33. Divergent

34. Absolutely convergent

35. Divergent

36. Absolutely convergent

37. Absolutely convergent

38. Divergent

8.4 SOLUTIONS

E Click here for exercises.

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3}{n+4}$. $b_n=\frac{3}{n+4}>0$ and $b_{n+1}<b_n$ for all n; $\lim_{n\to\infty}b_n=0$ so the series converges by the Alternating Series Test.

2. $-5+\sum_{n=0}^{\infty}(-1)^{n-1}\frac{5}{3n+2}$. $b_n=\frac{5}{3n+2}$ is decreasing and positive for all n, and $\lim_{n\to\infty}\frac{5}{3n+2}=0$ so the series converges by the Alternating Series Test.

3. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$. $\lim_{n\to\infty}\frac{n}{n+1}=1$ so $\lim_{n\to\infty}(-1)^n\frac{n}{n+1}$ does not exist and the series diverges by the Test for Divergence.

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$. $b_n=\frac{1}{n^2}>0$ and $b_{n+1}<b_n$ for all n, and $\lim_{n\to\infty}\frac{1}{n^2}=0$, so the series converges by the Alternating Series Test.

5. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$. $b_n=\frac{1}{\sqrt{n+3}}$ is positive and decreasing, and $\lim_{n\to\infty}\frac{1}{\sqrt{n+3}}=0$, so the series converges by the Alternating Series Test.

6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{5n+1}$. $\lim_{n\to\infty}\frac{n}{5n+1}=\frac{1}{5}$ so $\lim_{n\to\infty}(-1)^{n+1}\frac{n}{5n+1}$ does not exist and the series diverges by the Test for Divergence.

7. $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n\ln n}$. $b_n=\frac{1}{n\ln n}$ is positive and decreasing for $n\geq 2$, and $\lim_{n\to\infty}\frac{1}{n\ln n}=0$ so the series converges by the Alternating Series Test.

8. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$. $b_n=\frac{n}{n^2+1}>0$ for all n.
$b_{n+1}<b_n \Leftrightarrow \frac{n+1}{(n+1)^2+1}<\frac{n}{n^2+1} \Leftrightarrow$
$(n+1)\left(n^2+1\right)<\left[(n+1)^2+1\right]n \Leftrightarrow$
$n^3+n^2+n+1<n^3+2n^2+2n \Leftrightarrow$
$0<n^2+n-1$, which is true for all $n\geq 1$. Also $\lim_{n\to\infty}\frac{n}{n^2+1}=\lim_{n\to\infty}\frac{1/n}{1+1/n^2}=0$. Therefore the series converges by the Alternating Series Test.

9. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n^2}{n^2+1}$. $\lim_{n\to\infty}\frac{n^2}{n^2+1}=1$, so $\lim_{n\to\infty}(-1)^n\frac{n^2}{n^2+1}$ does not exist. Thus the series diverges by the Test for Divergence.

10. $a_n=(-1)^n\frac{2n}{4n+1}$, so $|a_n|=\frac{2n}{4n+1}\to\frac{1}{2}$ as $n\to\infty$. Therefore, $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist) and the series $\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n+1}$ diverges by the Test for Divergence.

11. $a_n=(-1)^{n-1}\frac{2n^2}{4n^2+1}$, so $|a_n|=\frac{2n^2}{4n^2+1}\to\frac{1}{2}$ as $n\to\infty$. Therefore, $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist) and the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n^2}{4n^2+1}$ diverges by the Test for Divergence.

12. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+4}$. $b_n=\frac{\sqrt{n}}{n+4}>0$ for all n. Let $f(x)=\frac{\sqrt{x}}{x+4}$. Then $f'(x)=\frac{4-x}{2\sqrt{x}\,(x+4)^2}<0$ if $x>4$, so $\{b_n\}$ is decreasing after $n=4$. $\lim_{n\to\infty}\frac{\sqrt{n}}{n+4}=\lim_{n\to\infty}\frac{1}{\sqrt{n}+4/\sqrt{n}}=0$. So the series converges by the Alternating Series Test.

13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2^n}$. $b_n=\frac{n}{2^n}>0$ and $b_n\geq b_{n+1} \Leftrightarrow \frac{n}{2^n}\geq\frac{n+1}{2^{n+1}} \Leftrightarrow 2n\geq n+1 \Leftrightarrow n\geq 1$ which is certainly true. $\lim_{n\to\infty}(n/2^n)=0$ by l'Hospital's Rule, so the series converges by the Alternating Series Test.

14. $\frac{1}{\sqrt[3]{\ln n}}$ decreases and $\lim_{n\to\infty}\frac{1}{\sqrt[3]{\ln n}}=0$, so by the Alternating Series Test the series converges.

15. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}$. $b_5=\frac{1}{(2\cdot 5-1)!}=\frac{1}{362{,}880}<0.00001$, so $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}\approx\sum_{n=1}^{4}\frac{(-1)^{n-1}}{(2n-1)!}\approx 0.8415$.

16. $b_4=\frac{1}{(2\cdot 4)!}=\frac{1}{40{,}320}\approx 0.000025$ and $s_3=1-\frac{1}{2}+\frac{1}{24}-\frac{1}{720}\approx 0.54028$, so, correct to four decimal places, $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\approx 0.5403$.

17. $b_6 = \dfrac{1}{2^6 6!} = \dfrac{1}{46{,}080} \approx 0.000022 < 0.0001$, so $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \approx \sum_{n=0}^{5} \frac{(-1)^n}{2^n n!} \approx 0.6065.$

18. $b_8 = 1/8^6 \approx 0.0000038 < 0.00001$ and

$$s_7 = 1 - \tfrac{1}{64} + \tfrac{1}{729} - \tfrac{1}{4096} + \tfrac{1}{15{,}625} - \tfrac{1}{46{,}656} + \tfrac{1}{117{,}649} \approx 0.9855537$$

so correct to five decimal places, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6} \approx 0.98555.$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series ($p = \frac{3}{2} > 1$), so the given series is absolutely convergent.

20. Using the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}/(n+1)^3}{(-3)^n/n^3}\right| = 3 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = 3 > 1$$

so the series diverges.

21. Using the Ratio Test, $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}/(n+1)!}{(-3)^n/n!}\right| = 3 \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$, so the series is absolutely convergent.

22. $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ is absolutely convergent by the Ratio Test.

23. $\dfrac{1}{n^2+1} < \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p = 2 > 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges absolutely by the Comparison Test.

24. $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/[(2n+1)!]}{1/[(2n-1)!]} = \lim_{n\to\infty} \frac{1}{(2n+1)\,2n} = 0$$

so by the Ratio Test the series is absolutely convergent.

25. $\displaystyle\lim_{n\to\infty} \frac{2n}{3n-4} = \frac{2}{3}$, so $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n}{3n-4}$ diverges by the Test for Divergence.

26. $\displaystyle\lim_{n\to\infty} (-1)^n \frac{2^n}{n^2+1}$ does not exist, so $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2+1}$ diverges by the Test for Divergence.

27. $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}/[(n+1)\,3^{n+2}]}{2^n/(n3^{n+1})}\right| = \frac{2}{3} \lim_{n\to\infty} \frac{n}{n+1} = \frac{2}{3} < 1$$

so the series converges absolutely by the Ratio Test.

28. $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{5^n/[(n+2)^2\,4^{n+3}]}{5^{n-1}/[(n+1)^2\,4^{n+2}]}\right| = \frac{5}{4} \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^2 = \frac{5}{4} > 1$$

so the series diverges by the Ratio Test.

29. $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+2)\,5^{n+1}\Big/\Big[(n+1)\,3^{2(n+1)}\Big]}{(n+1)\,5^n/(n3^{2n})} = \lim_{n\to\infty} \frac{5n\,(n+2)}{9\,(n+1)^2} = \frac{5}{9} < 1$$

so the series converges absolutely by the Ratio Test.

30. $\left|\dfrac{\sin 2n}{n^2}\right| \le \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{n^2}$ converges absolutely by the Comparison Test.

31. $\dfrac{\arctan n}{n^3} < \dfrac{\pi/2}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{\pi/2}{n^3}$ converges ($p = 3 > 1$), so $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^3}$ converges absolutely by the Comparison Test.

32. $\left|\cos \dfrac{n\pi}{6}\right| \le 1$, so since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges ($p = \frac{3}{2} > 1$), the given series converges absolutely by the Comparison Test.

33. $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!/10^{n+1}}{n!/10^n} = \lim_{n\to\infty} \frac{n+1}{10} = \infty$$

so the series diverges by the Ratio Test.

34. $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\left[8-(n+1)^3\right]/[(n+1)!]}{(8-n^3)/(n!)}\right| = \lim_{n\to\infty} \frac{1}{n+1} \left|\frac{8-(n+1)^3}{8-n^3}\right| = 0 < 1$$

so the series converges absolutely by the Ratio Test.

35. $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}/5^{2n+5}}{n^n/5^{2n+3}} = \lim_{n\to\infty} \frac{1}{25} \left(\frac{n+1}{n}\right)^n (n+1) = \infty$$

so the series diverges by the Ratio Test.

36. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(n+1)^2/[(n+3)!]}{(-2)^n n^2/[(n+2)!]}\right|$$
$$= \lim_{n\to\infty}\frac{2(n+1)^2}{n^2(n+3)} = 0 < 1$$
so the series converges absolutely by the Ratio Test.

37. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+3)!/\left[(n+1)!10^{n+1}\right]}{(n+2)!/(n!10^n)}$$
$$= \frac{1}{10}\lim_{n\to\infty}\frac{n+3}{n+1} = \frac{1}{10} < 1$$
so the series converges absolutely by the Ratio Test.

38. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)(3n+1)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)(2n+3)}}{\dfrac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}}\right|$$
$$= \lim_{n\to\infty}\frac{3n+1}{2n+3} = \frac{3}{2} > 1$$
so the series diverges by the Ratio Test.