8.6 REPRESENTING FUNCTIONS AS POWER SERIES

A Click here for answers.

S Click here for solutions.

1–7 ■ Find a power series representation for the function and determine the interval of convergence.

1. $f(x) = \dfrac{x}{1-x}$

2. $f(x) = \dfrac{1}{4+x^2}$

3. $f(x) = \dfrac{1+x^2}{1-x^2}$

4. $f(x) = \dfrac{1}{1+4x^2}$

5. $f(x) = \dfrac{1}{x^4+16}$

6. $f(x) = \dfrac{x}{x-3}$

7. $f(x) = \dfrac{2}{3x+4}$

8–9 ■ Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

8. $f(x) = \dfrac{3x-2}{2x^2-3x+1}$

9. $f(x) = \dfrac{x}{x^2-3x+2}$

10–11 ■ Find a power series representation for the function and determine the radius of convergence.

10. $f(x) = \tan^{-1}(2x)$

11. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

12–14 ■ Evaluate the indefinite integral as a power series.

12. $\displaystyle\int \frac{1}{1+x^4}\,dx$

13. $\displaystyle\int \frac{x}{1+x^5}\,dx$

14. $\displaystyle\int \frac{\arctan x}{x}\,dx$

15–16 ■ Use a power series to approximate the definite integral to six decimal places.

15. $\displaystyle\int_0^{0.2} \frac{1}{1+x^4}\,dx$

16. $\displaystyle\int_0^{1/2} \tan^{-1}(x^2)\,dx$

8.6 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\sum_{n=1}^{\infty} x^n$, $(-1, 1)$

2. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}$, $(-2, 2)$

3. $1 + \sum_{n=1}^{\infty} 2x^{2n}$, $(-1, 1)$

4. $\sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$

5. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{4n+4}}$, $(-2, 2)$

6. $-\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$, $(-3, 3)$

7. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{2^{2n+1}}$, $\left(-\frac{4}{3}, \frac{4}{3}\right)$

8. $-\sum_{n=0}^{\infty} (2^n + 1) x^n$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$

9. $\sum_{n=0}^{\infty} \left(1 - 2^{-n}\right) x^n$, $(-1, 1)$

10. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$, $\frac{1}{2}$

11. $\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$, 1

12. $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$

13. $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+2}}{5n+2}$

14. $C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$

15. 0.199936

16. 0.041303

8.6 SOLUTIONS

E Click here for exercises.

"R" stands for "radius of convergence" and "I" stands for "interval of convergence" in this section.

1. $f(x) = \dfrac{x}{1-x} = x\left(\dfrac{1}{1-x}\right) = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$

$= \sum_{n=1}^{\infty} x^n$

with $R = 1$ and $I = (-1, 1)$.

2. $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\left(\dfrac{1}{1+x^2/4}\right)$

$= \dfrac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x^2}{4}\right)^n$ (using Exercise 3 from the text)

$= \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{4^{n+1}}$

with $\left|\dfrac{x^2}{4}\right| < 1 \iff x^2 < 4 \iff |x| < 2$, so $R = 2$ and $I = (-2, 2)$.

3. $f(x) = \dfrac{1+x^2}{1-x^2} = 1 + \dfrac{2x^2}{1-x^2} = 1 + 2x^2\sum_{n=0}^{\infty}(x^2)^n$

$= 1 + \sum_{n=0}^{\infty} 2x^{2n+2} = 1 + \sum_{n=1}^{\infty} 2x^{2n}$

with $|x^2| < 1 \iff |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \dfrac{1}{1+4x^2} = \sum_{n=0}^{\infty}(-1)^n(4x^2)^n$ (using Exercise 3 from the text)

$= \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}$

with $|4x^2| < 1$ so $x^2 < \frac{1}{4} \iff |x| < \frac{1}{2}$, and so $R = \frac{1}{2}$ and $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

5. $f(x) = \dfrac{1}{x^4+16} = \dfrac{1}{16}\left[\dfrac{1}{1+(x/2)^4}\right]$

$= \dfrac{1}{16}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x}{2}\right)^{4n} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{4n}}{2^{4n+4}}$

for $\left|\dfrac{x}{2}\right| < 1 \iff |x| < 2$ so, $R = 2$ and $I = (-2, 2)$.

6. $f(x) = \dfrac{x}{x-3} = 1 + \dfrac{3}{x-3} = 1 - \dfrac{1}{1-x/3}$

$= 1 - \sum_{n=0}^{\infty}\left(\dfrac{x}{3}\right)^n = -\sum_{n=1}^{\infty}\left(\dfrac{x}{3}\right)^n$

For convergence, $\dfrac{|x|}{3} < 1 \iff |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

Another Method:

$f(x) = \dfrac{x}{x-3} = -\dfrac{x}{3(1-x/3)} = -\dfrac{x}{3}\sum_{n=0}^{\infty}\left(\dfrac{x}{3}\right)^n$

$= -\sum_{n=0}^{\infty}\dfrac{x^{n+1}}{3^{n+1}} = -\sum_{n=1}^{\infty}\dfrac{x^n}{3^n}$

7. $f(x) = \dfrac{2}{3x+4} = \dfrac{1}{2}\left(\dfrac{1}{1+3x/4}\right)$

$= \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3x}{4}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n 3^n x^n}{2^{2n+1}}$

$\left|\dfrac{3x}{4}\right| < 1$ so $R = \frac{4}{3}$ and $I = \left(-\frac{4}{3}, \frac{4}{3}\right)$.

8. $\dfrac{3x-2}{2x^2-3x+1} = \dfrac{3x-2}{(2x-1)(x-1)} = \dfrac{A}{2x-1} + \dfrac{B}{x-1}$

$\iff A + 2B = 3$ and $-A - B = -2 \iff A = B = 1$, so

$f(x) = \dfrac{3x-2}{2x^2-3x+1} = \dfrac{1}{2x-1} + \dfrac{1}{x-1}$

$= -\sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty}x^n = -\sum_{n=0}^{\infty}(2^n+1)x^n$

with $R = \frac{1}{2}$. At $x = \pm\frac{1}{2}$, the series diverges by the Test for Divergence, so $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

9. $\dfrac{x}{x^2-3x+2} = \dfrac{x}{(x-2)(x-1)} = \dfrac{A}{x-2} + \dfrac{B}{x-1} \iff$

$A + B = 1$ and $-A - 2B = 0 \iff A = 2, B = -1$, so

$f(x) = \dfrac{x}{(x-2)(x-1)} = \dfrac{2}{x-2} - \dfrac{1}{x-1}$

$= -\dfrac{1}{1-x/2} + \dfrac{1}{1-x} = -\sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n + \sum_{n=0}^{\infty}x^n$

$= \sum_{n=0}^{\infty}(1-2^{-n})x^n$

with $R = 1$. At $x = \pm 1$, the series diverges by the Test for Divergence, so $I = (-1, 1)$.

10. $f(x) = \tan^{-1}2x = 2\displaystyle\int\dfrac{dx}{1+4x^2}$

$= 2\displaystyle\int\sum_{n=0}^{\infty}(-1)^n(4x^2)^n\,dx$

$= 2\displaystyle\int\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\,dx$

$= C + 2\sum_{n=0}^{\infty}\dfrac{(-1)^n 4^n x^{2n+1}}{2n+1}$

$= \sum_{n=0}^{\infty}\dfrac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$

for $|4x^2| < 1$ so $|x| < \frac{1}{2}$ and $R = \frac{1}{2}$.

11. $f(x) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x}$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n \right] dx$$

$$= \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} + C$$

But $f(0) = \ln 1 - \ln 1 = 0$, so $C = 0$ and we have

$f(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$.

12. $\int \frac{dx}{1+x^4} = \int \sum_{n=0}^{\infty} (-1)^n x^{4n}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$

with $R = 1$.

13. $\frac{1}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n x^{5n} \quad \Rightarrow$

$\frac{x}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n x^{5n+1} \quad \Rightarrow$

$\int \frac{x}{1+x^5}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+2}}{5n+2}$ with $R = 1$.

14. By Example 7, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, so

$$\int \frac{\arctan x}{x}\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}\, dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$$

with $R = 1$.

15. We use the representation

$\int \frac{dx}{1+x^4} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$ from

Problem 12 with $C = 0$. So

$$\int_0^{0.2} \frac{dx}{1+x^4} = \left[x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots \right]_0^{0.2}$$

$$= 0.2 - \frac{0.2^5}{5} + \frac{0.2^9}{9} - \frac{0.2^{13}}{13} + \cdots$$

Since the series is alternating, the error in the nth-order approximation is less than the first neglected term, by The Alternating Series Estimation Theorem. If we use only the first two terms of the series, then the error is at most $0.2^9/9 \approx 5.7 \times 10^{-8}$. So, to six decimal places,

$$\int_0^{0.2} \frac{dx}{1+x^4} \approx 0.2 - \frac{0.2^5}{5} = 0.199936.$$

16. Using Example 7 with x replaced by x^2, we have

$\int \tan^{-1}(x^2)\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$

so

$$\int_0^{1/2} \tan^{-1}(x^2)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105} + \frac{x^{19}}{171} - \cdots \right]_0^{1/2}$$

$$= \frac{0.5^3}{3} - \frac{0.5^7}{21} + \frac{0.5^{11}}{55} - \frac{0.5^{15}}{105} + \frac{0.5^{19}}{171} - \cdots$$

The series is alternating, so if we use only the first four terms of the series, then the error is at most $0.5^{19}/171 \approx 1.1 \times 10^{-8}$. So, to six decimal places,

$$\int_0^{1/2} \tan^{-1}(x^2)\, dx$$

$$\approx \tfrac{1}{3}(0.5)^3 - \tfrac{1}{21}(0.5)^7 + \tfrac{1}{55}(0.5)^{11} - \tfrac{1}{105}(0.5)^{15}$$

$$\approx 0.041303$$