8.7 TAYLOR AND MACLAURIN SERIES

A Click here for answers.

S Click here for solutions.

1–2 ▪ Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

1. $f(x) = \dfrac{1}{(1+x)^2}$

2. $f(x) = \dfrac{x}{1-x}$

3–6 ▪ Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

3. $f(x) = 1/x, \quad a = 1$

4. $f(x) = \sqrt{x}, \quad a = 4$

5. $f(x) = \sin x, \quad a = \pi/4$

6. $f(x) = \cos x, \quad a = -\pi/4$

7–13 ▪ Use a Maclaurin series derived in this section to obtain the Maclaurin series for the given function.

7. $f(x) = e^{3x}$

8. $f(x) = \sin 2x$

9. $f(x) = x^2 \cos x$

10. $f(x) = \cos(x^3)$

11. $f(x) = x \sin(x/2)$

12. $f(x) = xe^{-x}$

13. $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$

14–15 ▪ Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials on the same screen. What do you notice about the relationship between these polynomials and f?

14. $f(x) = 1/\sqrt{1+2x}$

15. $f(x) = (1+x)^{-3}$

16. Find the Maclaurin series for $\ln(1+x)$ and use it to calculate $\ln 1.1$ correct to five decimal places.

17–18 ▪ Evaluate the indefinite integral as an infinite series.

17. $\displaystyle\int \sin(x^2)\,dx$

18. $\displaystyle\int e^{x^3}\,dx$

19–20 ▪ Use series to approximate the definite integral correct to three decimal places.

19. $\displaystyle\int_0^1 \sin(x^2)\,dx$

20. $\displaystyle\int_0^{0.5} \cos(x^2)\,dx$

21. Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for

$$y = \frac{\ln(1-x)}{e^x}$$

22–24 ▪ Find the sum of the series.

22. $\displaystyle\sum_{n=2}^{\infty} \frac{x^{3n+1}}{n!}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)!}$

25. Show that $e^x > 1 + x$ for all $x > 0$.

26–31 ▪ Use the binomial series to expand the function as a power series. State the radius of convergence.

26. $\sqrt[3]{1+x^2}$

27. $\dfrac{x}{\sqrt{1-x}}$

28. $\dfrac{1}{\sqrt{2+x}}$

29. $\dfrac{x^2}{\sqrt{1-x^3}}$

30. $\left(\dfrac{x}{1-x}\right)^5$

31. $\sqrt[5]{x-1}$

32. (a) Expand $1/\sqrt{1+x}$ as a power series.
(b) Use part (a) to estimate $1/\sqrt{1.1}$ correct to three decimal places.

33. (a) Expand $\sqrt[3]{8+x}$ as a power series.
(b) Use part (a) to estimate $\sqrt[3]{8.2}$ correct to four decimal places.

8.7 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$, $R=1$

2. $\sum_{n=1}^{\infty} x^n$, $R=1$

3. $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$, $R=1$

4. $2+\dfrac{x-4}{4}+\sum_{n=2}^{\infty} \dfrac{(-1)^{n-1}\, 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{3n-1}n!}(x-4)^n$, $R=4$

5. $\dfrac{\sqrt{2}}{2}\sum_{n=0}^{\infty} (-1)^n \left[\dfrac{1}{(2n)!}\left(x-\dfrac{\pi}{4}\right)^{2n} + \dfrac{1}{(2n+1)!}\left(x-\dfrac{\pi}{4}\right)^{2n+1}\right]$, $R=\infty$

6. $\dfrac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \dfrac{(-1)^{n(n-1)/2}\left(x+\frac{\pi}{4}\right)^n}{n!}$, $R=\infty$

7. $\sum_{n=0}^{\infty} \dfrac{3^n x^n}{n!}$, $R=\infty$

8. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$, $R=\infty$

9. $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+2}}{(2n)!}$, $R=\infty$

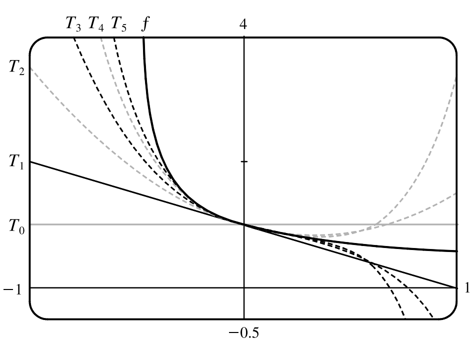
10. $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{6n}}{(2n)!}$, $R=\infty$

11. $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+2}}{(2n+1)!2^{2n+1}}$, $R=\infty$

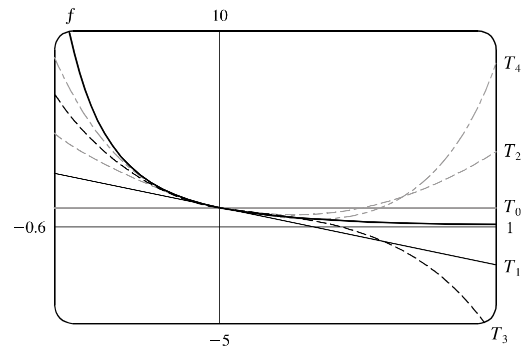
12. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^n}{(n-1)!}$, $R=\infty$

13. $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n+2)!}$, $R=\infty$

14. $\sum_{n=0}^{\infty} \dfrac{(-1)^n\, 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}x^n$, $R=\frac{1}{2}$.



15. $\sum_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2) x^n}{2}$, $R=1$



16. 0.09531

17. $C+\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$

18. $C+\sum_{n=0}^{\infty} \dfrac{x^{3n+1}}{(3n+1)\,n!}$

19. 0.310

20. 0.497

21. $-x+\dfrac{x^2}{2}-\dfrac{x^3}{3}+\cdots$

22. $x\left(e^{x^3}-1-x^3\right)$

23. e^x-1

24. $\dfrac{2}{x}\left(e^{x/2}-1\right)$

26. $1+\dfrac{x^2}{3}+\sum_{n=2}^{\infty} \dfrac{(-1)^{n-1}\cdot 2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)\,x^{2n}}{3^n\, n!}$, $R=1$

27. $x+\sum_{n=1}^{\infty} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}x^{n+1}$, $R=1$

28. $\dfrac{\sqrt{2}}{2}\left[1+\sum_{n=1}^{\infty} \dfrac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,x^n}{2^{2n}\cdot n!}\right]$, $R=2$

29. $x^2+\sum_{n=1}^{\infty} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,x^{3n+2}}{2^n\cdot n!}$, $R=1$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+4)!}{4! \cdot n!} x^{n+5}, R = 1$

31. $\displaystyle -1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{4 \cdot 9 \cdots (5n-6)\, x^n}{5^n \cdot n!}, R = 1$

32. (a) $\displaystyle 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^n$

(b) 0.953

33. (a) $\displaystyle 2\left[1 + \frac{x}{24} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 2 \cdot 5 \cdot \cdots \cdot (3n-4)\, x^n}{24^n \cdot n!}\right]$

(b) 2.0165

8.7 SOLUTIONS

E Click here for exercises.

1.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-2}$	1
1	$-2(1+x)^{-3}$	-2
2	$2\cdot 3(1+x)^{-4}$	$2\cdot 3$
3	$-2\cdot 3\cdot 4(1+x)^{-5}$	$-2\cdot 3\cdot 4$
4	$2\cdot 3\cdot 4\cdot 5(1+x)^{-6}$	$2\cdot 3\cdot 4\cdot 5$
$\cdots$	$\cdots$	$\cdots$

So $f^{(n)}(0) = (-1)^n (n+1)!$ and

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

If $a_n = (-1)^n (n+1) x^n$, then $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|$, so $R = 1$.

2.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x/(1-x)$	0
1	$(1-x)^{-2}$	1
2	$2(1-x)^{-3}$	2
3	$3\cdot 2(1-x)^{-4}$	$3\cdot 2$
4	$4\cdot 3\cdot 2(1-x)^{-5}$	$4\cdot 3\cdot 2$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(0) = n!$ except when $n = 0$, so

$\dfrac{x}{1-x} = \sum\limits_{n=1}^{\infty} \dfrac{n!}{n!} x^n = \sum\limits_{n=1}^{\infty} x^n$. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = |x| < 1$ for convergence, so $R = 1$.

3.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-3\cdot 2x^{-4}$	$-3\cdot 2$
4	$4\cdot 3\cdot 2x^{-5}$	$4\cdot 3\cdot 2$
$\cdots$	$\cdots$	$\cdots$

So $f^{(n)}(1) = (-1)^n n!$, and

$\dfrac{1}{x} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n n!}{n!}(x-1)^n = \sum\limits_{n=0}^{\infty} (-1)^n (x-1)^n$. If $a_n = (-1)^n (x-1)^n$ then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = |x-1| < 1$ for convergence, so $0 < x < 2$ and $R = 1$.

4.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{1/2}$	2
1	$\frac{1}{2}x^{-1/2}$	2^{-2}
2	$-\frac{1}{4}x^{-3/2}$	-2^{-5}
3	$\frac{3}{8}x^{-5/2}$	$3\cdot 2^{-8}$
4	$-\frac{15}{16}x^{-7/2}$	$-15\cdot 2^{-11}$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(4) = \dfrac{(-1)^{n-1}\, 1\cdot 3\cdot 5\cdot\dots\cdot(2n-3)}{2^{3n-1}}$ for $n \ge 2$, so

$$\sqrt{x} = 2 + \frac{x-4}{4} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1\cdot 3\cdot 5\cdot\dots\cdot(2n-3)}{2^{3n-1} n!}(x-4)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-4|}{8}\lim_{n\to\infty}\left(\frac{2n-1}{n+1}\right) = \frac{|x-4|}{4} < 1$$

for convergence, so $|x-4| < 4 \;\Rightarrow\; R = 4$.

5.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\sin x$	$\sqrt{2}/2$
1	$\cos x$	$\sqrt{2}/2$
2	$-\sin x$	$-\sqrt{2}/2$
3	$-\cos x$	$-\sqrt{2}/2$
4	$\sin x$	$\sqrt{2}/2$
$\cdots$	$\cdots$	$\cdots$

$$\begin{aligned}
\sin x &= f\left(\tfrac{\pi}{4}\right) + f'\left(\tfrac{\pi}{4}\right)\left(x-\tfrac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\tfrac{\pi}{4}\right)^2 \\
&\quad + \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}\left(x-\tfrac{\pi}{4}\right)^4 + \cdots \\
&= \tfrac{\sqrt{2}}{2}\left[1 + \left(x-\tfrac{\pi}{4}\right) - \tfrac{1}{2!}\left(x-\tfrac{\pi}{4}\right)^2 \right. \\
&\qquad \left. - \tfrac{1}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \tfrac{1}{4!}\left(x-\tfrac{\pi}{4}\right)^4 + \cdots\right] \\
&= \tfrac{\sqrt{2}}{2}\left[1 - \tfrac{1}{2!}\left(x-\tfrac{\pi}{4}\right)^2 + \tfrac{1}{4!}\left(x-\tfrac{\pi}{4}\right)^4 - \cdots\right] \\
&\qquad + \tfrac{\sqrt{2}}{2}\left[\left(x-\tfrac{\pi}{4}\right) - \tfrac{1}{3!}\left(x-\tfrac{\pi}{4}\right)^3 + \cdots\right] \\
&= \tfrac{\sqrt{2}}{2}\sum_{n=0}^{\infty}(-1)^n\left[\tfrac{1}{(2n)!}\left(x-\tfrac{\pi}{4}\right)^{2n} \right. \\
&\qquad \left. + \tfrac{1}{(2n+1)!}\left(x-\tfrac{\pi}{4}\right)^{2n+1}\right]
\end{aligned}$$

The series can also be written in the more elegant form

$\sin x = \dfrac{\sqrt{2}}{2}\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n(n-1)/2}\left(x-\frac{\pi}{4}\right)^n}{n!}$. If $a_n = \dfrac{(-1)^{n(n-1)/2}\left(x-\frac{\pi}{4}\right)^n}{n!}$, then

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{\left|x-\frac{\pi}{4}\right|}{n+1} = 0 < 1$ for all x, so $R = \infty$.

6.

n	$f^{(n)}(x)$	$f^{(n)}\left(-\frac{\pi}{4}\right)$
0	$\cos x$	$\frac{\sqrt{2}}{2}$
1	$-\sin x$	$\frac{\sqrt{2}}{2}$
2	$-\cos x$	$-\frac{\sqrt{2}}{2}$
3	$\sin x$	$-\frac{\sqrt{2}}{2}$
4	$\cos x$	$\frac{\sqrt{2}}{2}$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}\left(-\frac{\pi}{4}\right) = (-1)^{n(n-1)/2}\,\frac{\sqrt{2}}{2}$, so

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(-\frac{\pi}{4}\right)}{n!}\left(x+\frac{\pi}{4}\right)^n = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n(n-1)/2}\left(x+\frac{\pi}{4}\right)^n}{n!}$$

with $R = \infty$ by the Ratio Test (as in Problem 5).

7. $e^{3x} = \displaystyle\sum_{n=0}^{\infty}\frac{(3x)^n}{n!} = \sum_{n=0}^{\infty}\frac{3^n x^n}{n!}$, with $R = \infty$.

8. $\sin 2x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}x^{2n+1}}{(2n+1)!}$,

$R = \infty$

9. $x^2\cos x = x^2\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n)!}$,

$R = \infty$

10. $\cos\left(x^3\right) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n \left(x^3\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n}}{(2n)!}$, $R = \infty$.

11.
$$x\sin\left(\frac{x}{2}\right) = x\sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+1)!2^{2n+1}}$$
with $R = \infty$.

12.
$$xe^{-x} = x\sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{(n-1)!},\ R = \infty.$$

13.
$$\frac{1-\cos x}{x^2} = x^{-2}\left[1 - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right] = x^{-2}\left[-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-2}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+2)!}$$

since the series is equal to $\frac{1}{2}$ when $x = 0$; $R = \infty$.

14.

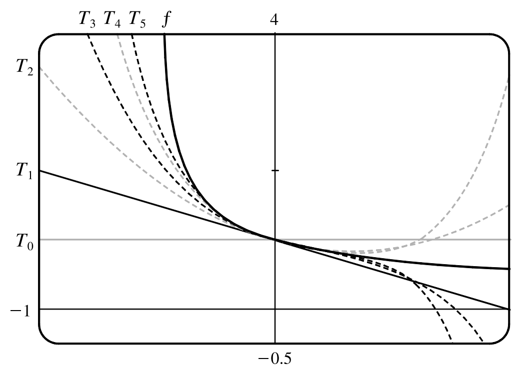
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+2x)^{-1/2}$	1
1	$-\frac{1}{2}(1+2x)^{-3/2}(2)$	-1
2	$\frac{3}{2}(1+2x)^{-5/2}(2)$	3
3	$-3\cdot\frac{5}{2}(1+2x)^{-7/2}(2)$	$-3\cdot 5$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(0) = (-1)^n\, 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)$, so

$$(1+2x)^{-1/2} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}x^n$$

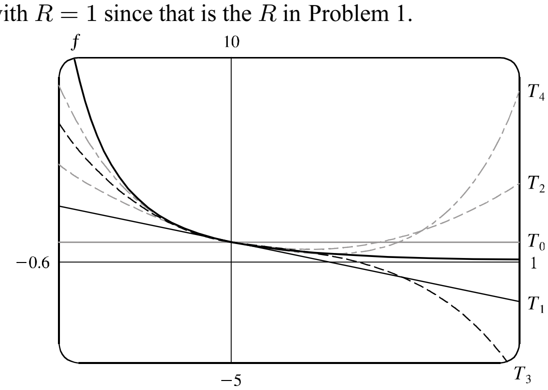
$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2n+1}{n+1}|x| = 2|x| < 1$ for convergence, so $R = \frac{1}{2}$.

Another method: Use Exercise 33 from the text and differentiate.



15.
$$f(x) = (1+x)^{-3} = -\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right] = -\frac{1}{2}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right] \quad \left[\begin{array}{l}\text{from}\\ \text{Problem 1}\end{array}\right]$$
$$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n n(n+1)x^{n-1} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)x^n}{2}$$

with $R = 1$ since that is the R in Problem 1.



16. $\ln(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

with $C = 0$ and $R = 1$, so $\ln(1.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (0.1)^n}{n}$.

This is an alternating series with

$b_5 = \frac{(0.1)^5}{5} = 0.000002$, so to five decimal places,

$\ln(1.1) \approx \sum_{n=1}^{4} \frac{(-1)^{n-1} (0.1)^n}{n} \approx 0.09531$.

17. $\int \sin(x^2) \, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \, dx$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \, dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$$

18. $\int e^{x^3} \, dx = \int \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} \, dx = C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)\, n!}$ with $R = \infty$.

19. Using our series from Problem 17, we get

$$\int_0^1 \sin(x^2) \, dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}$$

and $|c_3| = \frac{1}{75{,}600} < 0.000014$, so by the Alternating Series Estimation Theorem, we have

$\sum_{n=0}^{2} \frac{(-1)^n}{(4n+3)(2n+1)!} = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \approx 0.310$ (correct to three decimal places).

20. $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}$, so

$$\int_0^{0.5} \cos(x^2) \, dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \, dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} \right]_0^{0.5}$$

$$= 0.5 - \frac{(0.5)^5}{5 \cdot 2!} + \frac{(0.5)^9}{9 \cdot 4!} - \cdots$$

but $\frac{(0.5)^9}{9 \cdot 4!} \approx 0.000009$, so by the Alternating Series Estimation Theorem,

$\int_0^{0.5} \cos(x^2) \, dx \approx 0.5 - \frac{(0.5)^5}{5 \cdot 2!} \approx 0.497$ (correct to three decimal places).

21.

$$\begin{array}{r|l}
 & -x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \cdots \\
\hline
1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots & -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \cdots \\
 & -x - x^2 - \tfrac{1}{2}x^3 - \cdots \\
\hline
 & \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \cdots \\
 & \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \cdots \\
\hline
 & -\tfrac{1}{3}x^3 + \cdots \\
 & -\tfrac{1}{3}x^3 + \cdots \\
\hline
 & \cdots
\end{array}$$

From Example 6 in Section 8.6, we have $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$, $|x| < 1$. Therefore, $y = \frac{\ln(1-x)}{e^x} = \frac{-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots}{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots}$. So by the long division above, $\frac{\ln(1-x)}{e^x} = -x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots$, $|x| < 1$.

22. $\sum_{n=2}^{\infty} \frac{x^{3n+1}}{n!} = x \sum_{n=2}^{\infty} \frac{(x^3)^n}{n!} = x \left[\sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} - 1 - x^3 \right]$

$$= x \left(e^{x^3} - 1 - x^3 \right) \text{ by (11)}$$

23. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) - 1$$

$$= e^x - 1 \text{ by (11)}$$

24. $\sum_{n=0}^{\infty} \frac{x^n}{2^n (n+1)!} = \sum_{n=0}^{\infty} \frac{(x/2)^n}{(n+1)!} = \frac{2}{x} \sum_{n=0}^{\infty} \frac{(x/2)^n}{(n+1)!}$

$$= \frac{2}{x} \left[(x/2) + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \cdots \right]$$

$$= \frac{2}{x} \left(e^{x/2} - 1 \right)$$

25. By (12), $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, but for $x > 0$, all of the terms after the first two on the RHS are positive, so $e^x > 1 + x$ for $x > 0$.

26. $\left(1+x^2\right)^{1/3} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} x^{2n}$

$$= 1 + \frac{x^2}{3} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^4 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^6 + \cdots$$

$$= 1 + \frac{x^2}{3} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-4)\, x^{2n}}{3^n\, n!}$$

with $R = 1$.

27. $[1+(-x)]^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x)^n$

$$= 1 + \left(-\tfrac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2 + \cdots$$

$$= 1 + \frac{x}{2} + \frac{1\cdot 3}{2^2 2!}x^2 + \frac{1\cdot 3\cdot 5}{2^3 3!}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}x^4 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n n!} x^n$$

so $\dfrac{x}{\sqrt{1-x}} = x + \sum_{n=1}^{\infty} \dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n n!} x^{n+1}$ with

$R = 1$.

28. $(2+x)^{-1/2} = \dfrac{1}{\sqrt{2}}\left(1+\dfrac{x}{2}\right)^{-1/2}$

$$= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{x}{2}\right)^n$$

$$= \frac{\sqrt{2}}{2}\left[1 + \left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2 + \cdots\right]$$

$$= \frac{\sqrt{2}}{2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)\, x^n}{2^{2n}\cdot n!}\right]$$

with $|x/2| < 1$ so $|x| < 2$ and $R = 2$.

29. $\left[1+\left(-x^3\right)\right]^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-x^3\right)^n$

$$= 1 + \left(-\tfrac{1}{2}\right)\left(-x^3\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^3\right)^2 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)\, x^{3n}}{2^n\cdot n!}$$

so $\dfrac{x^2}{\sqrt{1-x^3}} = x^2 + \sum_{n=1}^{\infty} \dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)\, x^{3n+2}}{2^n\cdot n!}$

with $R = 1$.

30. $(1-x)^{-5} = 1 + (-5)(-x) + \dfrac{(-5)(-6)}{2!}(-x)^2 + \dfrac{(-5)(-6)(-7)}{3!}(-x)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{5\cdot 6\cdot 7\cdots (n+4)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+4)!}{4!\cdot n!} x^n$$

$\Rightarrow \quad \dfrac{x^5}{(1-x)^5} = \sum_{n=0}^{\infty} \dfrac{(n+4)!}{4!\cdot n!} x^{n+5}$ or

$\sum_{n=0}^{\infty} \dfrac{(n+1)(n+2)(n+3)(n+4)}{24} x^{n+5}$, with $R = 1$.

31. $\sqrt[5]{x-1} = -[1+(-x)]^{1/5} = -\sum_{n=0}^{\infty} \binom{\frac{1}{5}}{n} (-x)^n$

$$= -\left[1 + \tfrac{1}{5}(-x) + \frac{\left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)}{2!}(-x)^2 + \frac{\left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)}{3!}(-x)^3 + \cdots\right]$$

$$= -1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{4\cdot 9\cdots (5n-6)\, x^n}{5^n\cdot n!}$$ with $R = 1$.

32. (a) $(1+x)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)x + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n\cdot n!} x^n$$

(b) Take $x = 0.1$ in the above series.
$\frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}(0.1)^4 < 0.00003$, so
$\frac{1}{\sqrt{1.1}} \approx 1 - \frac{0.1}{2} + \frac{1\cdot 3}{2^2\cdot 2!}(0.1)^2 - \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}(0.1)^3 \approx 0.953$

33. (a) $(8+x)^{1/3} = 2\left(1+\dfrac{x}{8}\right)^{1/3} = 2\sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} \left(\dfrac{x}{8}\right)^n$

$$= 2\left[1 + \frac{1}{3}\left(\frac{x}{8}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{8}\right)^3 + \cdots\right]$$

$$= 2\left[1 + \frac{x}{24} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\cdot 2\cdot 5\cdot \cdots \cdot (3n-4)\, x^n}{24^n\cdot n!}\right]$$

(b) $(8+0.2)^{1/3} = 2\left[1 + \frac{0.2}{24} - \frac{(0.2)^2}{24^2} + \frac{2\cdot 5(0.2)^3}{24^3\cdot 3!} - \cdots\right]$

$$\approx 2\left[1 + \tfrac{0.2}{24} - \tfrac{(0.2)^2}{24^2}\right]$$

since $2\cdot \frac{2\cdot 5(0.2)^3}{24^3\cdot 3!} \approx 0.000002$, so $\sqrt[3]{8.2} \approx 2.0165$.