8.8 APPLICATIONS OF TAYLOR POLYNOMIALS

A Click here for answers.

S Click here for solutions.

1–6 ▪ Find the Taylor polynomial $T_n(x)$ for the function f at the number a. Graph f and T_n on the same screen.

1. $f(x) = \sqrt{x}, \quad a = 9, \quad n = 3$

2. $f(x) = 1/\sqrt[3]{x}, \quad a = 8, \quad n = 3$

3. $f(x) = \sec x, \quad a = \pi/3, \quad n = 3$

4. $f(x) = \tan x, \quad a = 0, \quad n = 4$

5. $f(x) = \tan x, \quad a = \pi/4, \quad n = 4$

6. $f(x) = e^x \sin x, \quad a = 0, \quad n = 3$

7–10 ▪

(a) Approximate f by a Taylor polynomial with degree n at the number a.

(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

(c) Check your result in part (b) by graphing $|R_n(x)|$.

7. $f(x) = \sin x, \quad a = \pi/4, \quad n = 5, \quad 0 \leqslant x \leqslant \pi/2$

8. $f(x) = \sqrt[3]{1 + x^2}, \quad a = 0, \quad n = 2, \quad |x| \leqslant 0.5$

9. $f(x) = x^{3/4}, \quad a = 16, \quad n = 3, \quad 15 \leqslant x \leqslant 17$

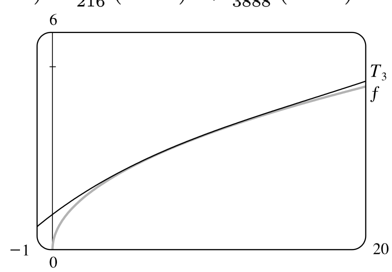
10. $f(x) = \ln x, \quad a = 4, \quad n = 3, \quad 3 \leqslant x \leqslant 5$

8.8 ANSWERS

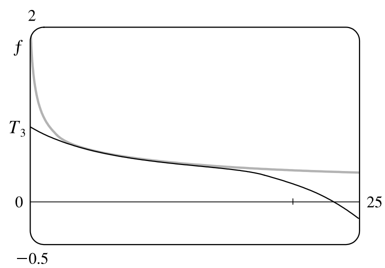
E Click here for exercises.

S Click here for solutions.

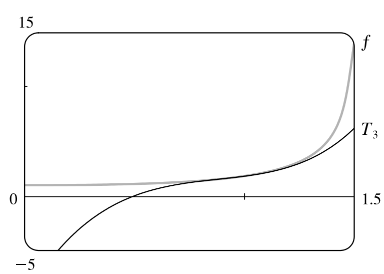
1. $3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3$



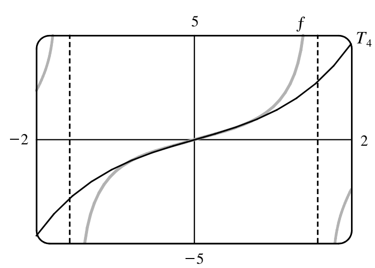
2. $\frac{1}{2} - \frac{1}{48}(x-8) + \frac{1}{576}(x-8)^2 - \frac{7}{41{,}472}(x-8)^3$



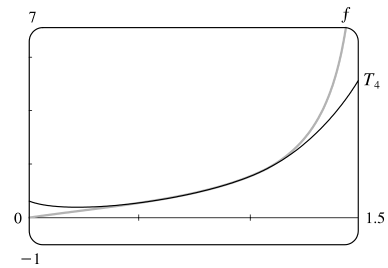
3. $2 + 2\sqrt{3}\left(x-\frac{\pi}{3}\right) + 7\left(x-\frac{\pi}{3}\right)^2 + \frac{23\sqrt{3}}{3}\left(x-\frac{\pi}{3}\right)^3$



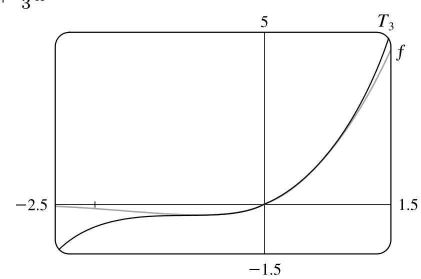
4. $x + \dfrac{x^3}{3}$



5. $1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x-\frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x-\frac{\pi}{4}\right)^4$



6. $x + x^2 + \frac{1}{3}x^3$



7. (a) $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2$
$- \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4 + \frac{\sqrt{2}}{240}\left(x-\frac{\pi}{4}\right)^5$

(b) 0.00033

8. (a) $1 + \frac{1}{3}x^2$

(b) 0.014895

9. (a) $8 + \frac{3}{8}(x-16) - \frac{3}{1024}(x-16)^2 + \frac{5}{65{,}536}(x-16)^3$

(b) 0.000003

10. (a) $\ln 4 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2 + \frac{1}{192}(x-4)^3$

(b) 0.0031

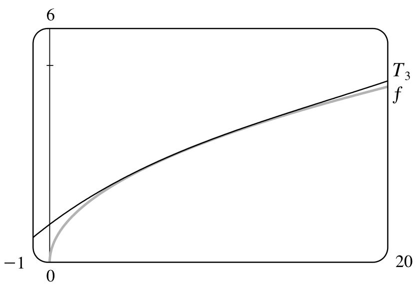
8.8 SOLUTIONS

E Click here for exercises.

1.

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{1/2}$	3
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{6}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{108}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{1}{648}$

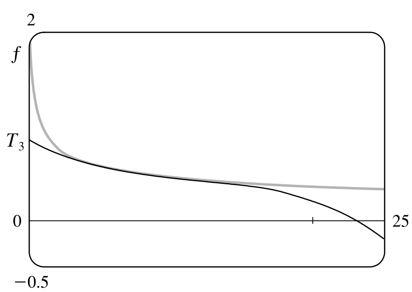
$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(9)}{n!}(x-9)^n$$
$$= 3 + \tfrac{1}{6}(x-9) - \tfrac{1}{216}(x-9)^2 + \tfrac{1}{3888}(x-9)^3$$



2.

n	$f^{(n)}(x)$	$f^{(n)}(8)$
0	$x^{-1/3}$	$\frac{1}{2}$
1	$-\frac{1}{3}x^{-4/3}$	$-\frac{1}{48}$
2	$\frac{4}{9}x^{-7/3}$	$\frac{1}{288}$
3	$-\frac{28}{27}x^{-10/3}$	$-\frac{7}{6912}$

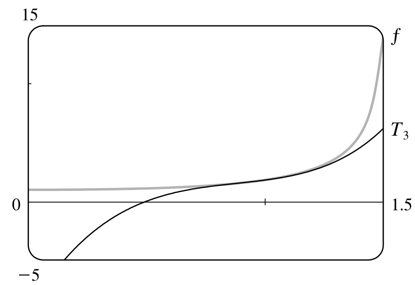
$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(8)}{n!}(x-8)^n$$
$$= \tfrac{1}{2} - \tfrac{1}{48}(x-8) + \tfrac{1}{576}(x-8)^2 - \tfrac{7}{41{,}472}(x-8)^3$$



3.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\sec x$	2
1	$\sec x \tan x$	$2\sqrt{3}$
2	$\sec x \tan^2 x + \sec^3$	14
3	$\sec x \tan^3 x + 5\sec^3 x \tan x$	$46\sqrt{3}$

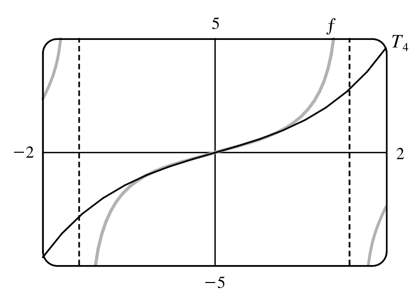
$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}\left(\frac{\pi}{3}\right)}{n!}\left(x-\frac{\pi}{3}\right)^n$$
$$= 2 + 2\sqrt{3}\left(x-\tfrac{\pi}{3}\right) + 7\left(x-\tfrac{\pi}{3}\right)^2 + \tfrac{23\sqrt{3}}{3}\left(x-\tfrac{\pi}{3}\right)^3$$



4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	0

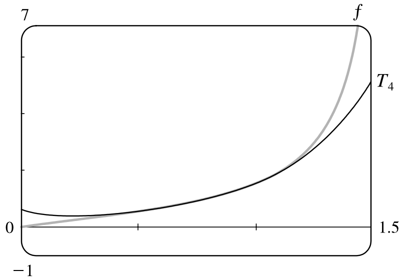
$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!}x^n = x + \frac{2x^3}{3!} = x + \frac{x^3}{3}$$



5.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\tan x$	1
1	$\sec^2 x$	2
2	$2\sec^2 x \tan x$	4
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	16
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	80

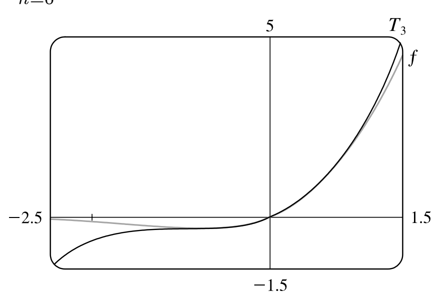
$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!}\left(x - \tfrac{\pi}{4}\right)^n$$
$$= 1 + 2\left(x - \tfrac{\pi}{4}\right) + 2\left(x - \tfrac{\pi}{4}\right)^2 + \tfrac{8}{3}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{10}{3}\left(x - \tfrac{\pi}{4}\right)^4$$



6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$e^x \sin x$	0
1	$e^x(\sin x + \cos x)$	1
2	$2e^x \cos x$	2
3	$2e^x(\cos x - \sin x)$	2

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = x + x^2 + \tfrac{1}{3}x^3$$



7.

$f(x) = \sin x$ $\qquad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$ $\qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x$ $\qquad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

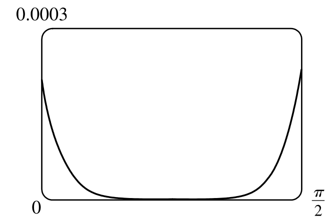
$f^{(5)}(x) = \cos x$ $\qquad f^{(5)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f^{(6)}(x) = -\sin x$

(a) $\sin x \approx T_5(x)$
$$= \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\left(x - \tfrac{\pi}{4}\right) - \tfrac{\sqrt{2}}{4}\left(x - \tfrac{\pi}{4}\right)^2 - \tfrac{\sqrt{2}}{12}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{\sqrt{2}}{48}\left(x - \tfrac{\pi}{4}\right)^4 + \tfrac{\sqrt{2}}{240}\left(x - \tfrac{\pi}{4}\right)^5$$

(b) $|R_5(x)| \le \frac{M}{6!}\left|x - \frac{\pi}{4}\right|^6$, where $\left|f^{(6)}(x)\right| \le M$. Now $0 \le x \le \frac{\pi}{2} \;\Rightarrow\; \left(x - \frac{\pi}{4}\right)^6 \le \left(\frac{\pi}{4}\right)^6$, and letting $x = \frac{\pi}{2}$ gives $M = 1$, so $|R_5(x)| \le \frac{1}{6!}\left(\frac{\pi}{4}\right)^6 = \frac{1}{720}\left(\frac{\pi}{4}\right)^6 \approx 0.00033$.

(c)



From the graph of $|R_5(x)| = |\sin x - T_5(x)|$, it seems that the error is less than 0.00026 on $\left[0, \frac{\pi}{2}\right]$.

8.

$f(x) = \left(1 + x^2\right)^{1/3}$ $\qquad f(0) = 1$

$f'(x) = \frac{2}{3}x\left(1 + x^2\right)^{-2/3}$ $\qquad f'(0) = 0$

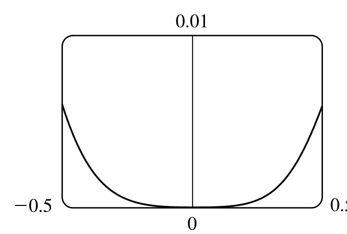
$f''(x) = \frac{2}{3}\left(1 - \frac{1}{3}x^2\right)\left(1 + x^2\right)^{-5/3}$ $\qquad f''(0) = \frac{2}{3}$

$f'''(x) = \dfrac{8x^3 - 72x}{27\left(1 + x^2\right)^{8/3}}$

(a) $\sqrt[3]{1 + x^2} \approx T_2(x) = 1 + \frac{1}{3}x^2$

(b) $|R_2(x)| \le \frac{M}{3!}|x|^3$, where $|f'''(x)| \le M$. By examining a graph of $|f'''(x)|$, we see that its maximum is approximately 0.71495314. Thus, $|R_2(x)| \le \frac{0.71495314}{3!}(0.5)^3 \approx 0.014895$.

(c)



It seems that the error is less than 0.0061 on $[-0.5, 0.5]$.

9.

$f(x) = x^{3/4}$ $\qquad f(16) = 8$

$f'(x) = \frac{3}{4}x^{-1/4}$ $\qquad f'(16) = \frac{3}{8}$

$f''(x) = -\frac{3}{16}x^{-5/4}$ $\qquad f''(16) = -\frac{3}{512}$

$f'''(x) = \frac{15}{64}x^{-9/4}$ $\qquad f'''(16) = \frac{15}{32{,}768}$

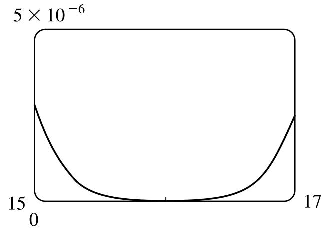
$f^{(4)}(x) = -\frac{135}{256}x^{-13/4}$

(a) $x^{3/4} \approx T_3(x)$
$$= 8 + \tfrac{3}{8}(x - 16) - \tfrac{3}{1024}(x - 16)^2 + \tfrac{5}{65{,}536}(x - 16)^3$$

(b) $|R_3(x)| \le \frac{M}{4!}|x-16|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $15 \le x \le 17 \;\Rightarrow\; |x-16|^4 \le 1^4 = 1$, and letting $x = 15$ to minimize the denominator of $f^{(4)}(x)$ gives

$$|R_3(x)| \le \frac{135 \Big/ \left[256\,(15)^{13/4}\right]}{4!}(1) \approx 0.000003.$$

(c)



It appears that the error is less than 3×10^{-6} on $[15, 17]$.

10.

$f(x) = \ln x$	$f(4) = \ln 4$
$f'(x) = x^{-1}$	$f'(4) = \frac{1}{4}$
$f''(x) = -x^{-2}$	$f''(4) = -\frac{1}{16}$
$f'''(x) = 2x^{-3}$	$f'''(4) = \frac{1}{32}$
$f^{(4)}(x) = -6x^{-4}$	

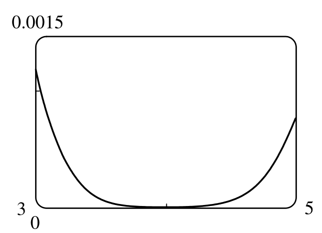
(a) $\ln x \approx T_3(x)$

$$= \ln 4 + \tfrac{1}{4}(x-4) - \tfrac{1}{32}(x-4)^2 + \tfrac{1}{192}(x-4)^3$$

(b) $|R_3(x)| \le \frac{M}{4!}|x-4|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $3 \le x \le 5 \;\Rightarrow\; (x-4)^4 \le 1^4 = 1$, and letting $x = 3$ gives $M = 6/3^4$, so

$$|R_3(x)| \le \frac{6}{4!\,3^4} \cdot 1 = \frac{1}{324} \approx 0.0031.$$

(c)



From the graph of $|R_3(x)| = |\ln x - T_3(x)|$, it appears that the error is less than 0.0013 on $[3, 5]$.