10.1 THREE-DIMENSIONAL COORDINATE SYSTEMS

A Click here for answers.

S Click here for solutions.

1–4 ▪ Draw a rectangular box that has P and Q as opposite vertices and has its face parallel to the coordinate planes. Then find (a) the coordinates of the other six vertices of the box and (b) the length of the diagonal of the box.

1. $P(0, 0, 0)$, $Q(2, 3, 5)$

2. $P(0, 0, 0)$, $Q(-4, -1, 2)$

3. $P(1, 1, 2)$, $Q(3, 4, 5)$

4. $P(4, 3, 0)$, $Q(1, 6, -4)$

5. Sketch the points $(3, 0, 1)$, $(-1, 0, 3)$, $(0, 4, -2)$, and $(1, 1, 0)$ on a single set of coordinate axes.

6–9 ▪ Find the lengths of the sides of the triangle ABC and determine whether the triangle is isosceles, a right triangle, both, or neither.

6. $A(3, -4, 1)$, $B(5, -3, 0)$, $C(6, -7, 4)$

7. $A(2, 1, 0)$, $B(3, 3, 4)$, $C(5, 4, 3)$

8. $A(5, 5, 1)$, $B(3, 3, 2)$, $C(1, 4, 4)$

9. $A(-2, 6, 1)$, $B(5, 4, -3)$, $C(2, -6, 4)$

10. Find an equation of the sphere with center $(0, 1, -1)$ and radius 4. What is the intersection of this sphere with the yz-plane?

11–13 ▪ Find the equation of the sphere with center C and radius r.

11. $C(-1, 2, 4)$, $r = \frac{1}{2}$

12. $C(-6, -1, 2)$, $r = 2\sqrt{3}$

13. $C(1, 2, -3)$, $r = 7$

14–19 ▪ Show that the equation represents a sphere, and find its center and radius.

14. $x^2 + y^2 + z^2 + 2x + 8y - 4z = 28$

15. $2x^2 + 2y^2 + 2z^2 + 4y - 2z = 1$

16. $x^2 + y^2 + z^2 = 6x + 4y + 10z$

17. $x^2 + y^2 + z^2 + x - 2y + 6z - 2 = 0$

18. $x^2 + y^2 + z^2 = x$

19. $x^2 + y^2 + z^2 + ax + by + cz + d = 0$, where $a^2 + b^2 + c^2 > 4d$

20. Find an equation of the sphere that has center $(1, 2, 3)$ and passes through the point $(-1, 1, -2)$.

21–28 ▪ Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality.

21. $x = 9$

22. $z = -8$

23. $y > 2$

24. $z \leqslant 0$

25. $|z| \leqslant 2$

26. $z = x$

27. $xy = 0$

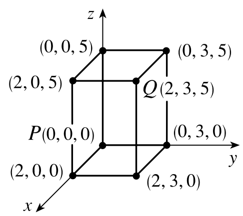
28. $xy = 1$

10.1 ANSWERS

E Click here for exercises.

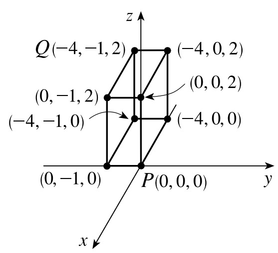
S Click here for solutions.

1. (a)



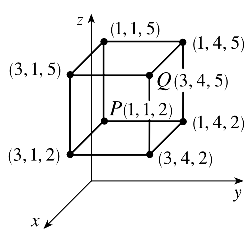
(b) $\sqrt{38}$

2. (a)



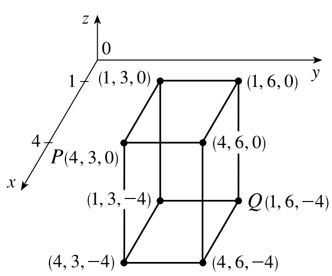
(b) $\sqrt{21}$

3. (a)



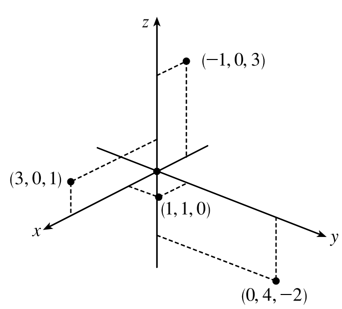
(b) $\sqrt{22}$

4. (a)



(b) $\sqrt{34}$

5.



6. $|AB| = \sqrt{6}$, $|BC| = \sqrt{33}$, $|CA| = 3\sqrt{3}$; right triangle

7. $|AB| = \sqrt{21}$, $|BC| = \sqrt{6}$, $|CA| = 3\sqrt{3}$; right triangle

8. $|AB| = 3$, $|BC| = 3$, $|CA| = \sqrt{26}$; isosceles

9. $|AB| = \sqrt{69}$, $|BC| = \sqrt{158}$, $|CA| = 13$; neither

10. $x^2 + (y-1)^2 + (z+1)^2 = 16$; $(y-1)^2 + (z+1)^2 = 16$, $x = 0$

11. $(x+1)^2 + (y-2)^2 + (z-4)^2 = \frac{1}{4}$

12. $(x+6)^2 + (y+1)^2 + (z-2)^2 = 12$

13. $(x-1)^2 + (y-2)^2 + (z+3)^2 = 49$

14. $(-1, -4, 2)$, 7

15. $\left(0, -1, \frac{1}{2}\right)$, $\frac{\sqrt{7}}{2}$

16. $(3, 2, 5)$, $\sqrt{38}$

17. $\left(-\frac{1}{2}, 1, -3\right)$, $\frac{7}{2}$

18. $\left(\frac{1}{2}, 0, 0\right)$, $\frac{1}{2}$

19. $\left(-\frac{1}{2}a, -\frac{1}{2}b, -\frac{1}{2}c\right)$, $\sqrt{\frac{1}{4}\left(a^2+b^2+c^2\right) - d}$

20. $(x-1)^2 + (y-2)^2 + (z-3)^2 = 30$

21. A plane parallel to the yz-plane and 9 units in front of it

22. A plane parallel to the xy-plane and 8 units below it

23. A half-space containing all points to the right of the plane $y = 2$

24. A half-space containing all points on and below the xy-plane

25. All points on and between the two horizontal planes $z = 2$ and $z = -2$

26. A plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$

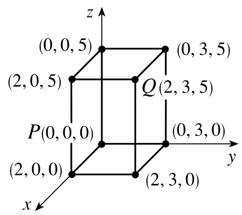
27. The two planes $x = 0$ and $y = 0$

28. A hyperbolic cylinder

10.1 SOLUTIONS

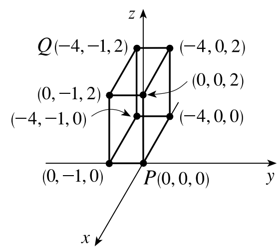
E Click here for exercises.

1. (a)



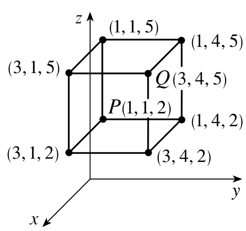
(b) $|PQ| = \sqrt{(2-0)^2 + (3-0)^2 + (5-0)^2} = \sqrt{38}$

2. (a)



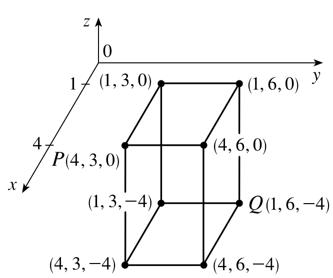
(b) $|PQ| = \sqrt{(-4-0)^2 + (-1-0)^2 + (2-0)^2} = \sqrt{21}$

3. (a)



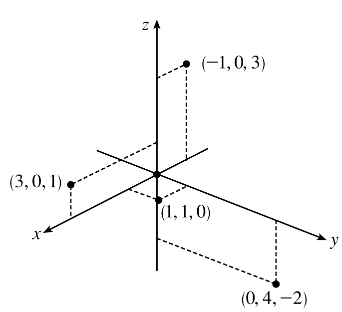
(b) $|PQ| = \sqrt{(3-1)^2 + (4-1)^2 + (5-2)^2} = \sqrt{22}$

4. (a)



(b) $|PQ| = \sqrt{(1-4)^2 + (6-3)^2 + (-4-0)^2} = \sqrt{34}$

5.



6. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|AB| = \sqrt{(5-3)^2 + [-3-(-4)]^2 + (0-1)^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$|BC| = \sqrt{(6-5)^2 + [-7-(-3)]^2 + (4-0)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$|CA| = \sqrt{(3-6)^2 + [-4-(-7)]^2 + (1-4)^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$$

Since the Pythagorean Theorem is satisfied by $|AB|^2 + |CA|^2 = |BC|^2$, ABC is a right triangle. ABC is not isosceles, as no two sides have the same length.

7. $|AB| = \sqrt{(3-2)^2 + (3-1)^2 + (4-0)^2} = \sqrt{21}$

$|BC| = \sqrt{(5-3)^2 + (4-3)^2 + (3-4)^2} = \sqrt{6}$

$|CA| = \sqrt{(5-2)^2 + (4-1)^2 + (3-0)^2} = \sqrt{27} = 3\sqrt{3}$

Since no two of the sides are equal in length the triangle isn't isosceles. But $|AB|^2 + |BC|^2 = 27 = |CA|^2$, so the triangle is a right triangle.

8. $|AB| = \sqrt{(3-5)^2 + (3-5)^2 + (2-1)^2} = \sqrt{9} = 3$

$|BC| = \sqrt{(1-3)^2 + (4-3)^2 + (4-2)^2} = \sqrt{9} = 3$

$|CA| = \sqrt{(5-1)^2 + (5-4)^2 + (1-4)^2} = \sqrt{26}$

Since $|AB| = |BC|$ the triangle is isosceles. But the sum of the squares of the lengths of the shorter sides doesn't equal the square of the length of the longest side, so the triangle isn't a right triangle.

9. $$|AB| = \sqrt{[5-(-2)]^2 + (4-6)^2 + (-3-1)^2} = \sqrt{49+4+16} = \sqrt{69}$$

$$|BC| = \sqrt{(2-5)^2 + (-6+4)^2 + [4-(-3)]^2} = \sqrt{9+100+49} = \sqrt{158}$$

$$|CA| = \sqrt{(-2-2)^2 + [6-(-6)]^2 + (1-4)^2} = \sqrt{16+144+9} = \sqrt{169} = 13$$

Since no two sides are of equal length and since $|AB|^2 + |BC|^2 \neq |CA|^2$ the triangle is neither isosceles nor a right triangle.

10. An equation of the sphere with center $(0, 1, -1)$ and radius 4 is $(x-0)^2 + (y-1)^2 + [z-(-1)]^2 = 4^2$ or $x^2 + (y-1)^2 + (z+1)^2 = 16$. The intersection of this sphere with the yz-plane is the set of points on the sphere whose x-coordinate is 0. Putting $x = 0$ in the equation, we have $(y-1)^2 + (z+1)^2 = 16$, $x = 0$, which represents a circle in the yz-plane with center $(0, 1, -1)$ and radius 4.

11. $[x-(-1)]^2 + (y-2)^2 + (z-4)^2 = \left(\frac{1}{2}\right)^2$ or $(x+1)^2 + (y-2)^2 + (z-4)^2 = \frac{1}{4}$

12. $(x+6)^2 + (y+1)^2 + (z-2)^2 = 12$

13. $(x-1)^2 + (y-2)^2 + (z+3)^2 = 49$

14. Completing squares in the equation gives

$$\left(x^2 + 2x + 1\right) + \left(y^2 + 8y + 16\right) + \left(z^2 - 4z + 4\right) = 28 + 1 + 16 + 4 \quad \Rightarrow$$

$(x+1)^2 + (y+4)^2 + (z-2)^2 = 49$, which we recognize as an equation of a sphere with center $(-1, -4, 2)$ and radius 7.

15. Completing squares in the equation gives

$$2x^2 + 2\left(y^2 + 2y + 1\right) + 2\left(z^2 - z + \tfrac{1}{4}\right) = 1 + 2 + \tfrac{1}{2} = \tfrac{7}{2} \quad \Rightarrow$$

$(x-0)^2 + (y+1)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{7}{4}$, which we recognize as an equation of a sphere with center $\left(0, -1, \frac{1}{2}\right)$ and radius $\frac{\sqrt{7}}{2}$.

16. $$\left(x^2 - 6x + 9\right) + \left(y^2 - 4y + 4\right) + \left(z^2 - 10z + 25\right) = 9 + 4 + 25 \quad \Rightarrow$$

$(x-3)^2 + (y-2)^2 + (z-5)^2 = 38 \quad \Rightarrow \quad C\,(3, 2, 5)$, and $r = \sqrt{38}$.

17. $$\left(x^2 + x + \tfrac{1}{4}\right) + \left(y^2 - 2y + 1\right) + \left(z^2 + 6z + 9\right) = 2 + \tfrac{1}{4} + 1 + 9 \quad \Rightarrow$$

$\left(x + \frac{1}{2}\right)^2 + (y-1)^2 + (z+3)^2 = \frac{49}{4} \quad \Rightarrow$
$C\left(-\frac{1}{2}, 1, -3\right)$, and $r = \frac{7}{2}$.

18. $\left(x^2 - x + \frac{1}{4}\right) + y^2 + z^2 = 0 + \frac{1}{4} \quad \Rightarrow$
$\left(x - \frac{1}{2}\right)^2 + (y-0)^2 + (z-0)^2 = \frac{1}{4} \quad \Rightarrow \quad C\left(\frac{1}{2}, 0, 0\right)$, $r = \frac{1}{2}$.

19. $$\left(x^2 + ax + \tfrac{1}{4}a^2\right) + \left(y^2 + by + \tfrac{1}{4}b^2\right) + \left(z^2 + cz + \tfrac{1}{4}c^2\right) = -d + \tfrac{1}{4}\left(a^2 + b^2 + c^2\right) \quad \Rightarrow$$

$\left(x + \frac{1}{2}a\right)^2 + \left(y + \frac{1}{2}b\right)^2 + \left(z + \frac{1}{2}c\right)^2 = \frac{1}{4}\left(a^2 + b^2 + c^2\right) - d$
$\Rightarrow \quad C\left(-\frac{1}{2}a, -\frac{1}{2}b, -\frac{1}{2}c\right)$, and $r = \sqrt{\frac{1}{4}\left(a^2 + b^2 + c^2\right) - d}$.

20. r^2 is equal to the square of the distance between the center of the sphere, $(1, 2, 3)$, and the given point $(-1, 1, -2)$. That is, $r^2 = (-1-1)^2 + (1-2)^2 + (-2-3)^2 = 30$. Therefore, an equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 30$.

21. The equation $x = 9$ represents a plane parallel to the yz-plane and 9 units in front of it.

22. The equation $z = -8$ represents a plane parallel to the xy-plane and 8 units below it.

23. The inequality $y > 2$ represents a half-space containing all points to the right of the plane $y = 2$.

24. The inequality $z \leq 0$ represents a half-space containing all points on and below the xy-plane.

25. The inequality $|z| \leq 2$ is equivalent to $-2 \leq z \leq 2$, so it represents all points on and between the two horizontal planes $z = 2$ and $z = -2$.

26. The equation $z = x$ represents a plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$.

27. Since either x or y must be zero, the region consists of the two perpendicular planes $x = 0$ (the yz-plane) and $y = 0$ (the xz-plane).

28. In the xy-plane the equation $xy = 1$ represents a hyperbola with center at the origin. Since z can assume any value, the region in $\mathbb{R}^3$ is a hyperbolic cylinder.