10.3 THE DOT PRODUCT

A Click here for answers.

S Click here for solutions.

1–7 ▪ Find $\mathbf{a} \cdot \mathbf{b}$.

1. $\mathbf{a} = \langle 2, 5 \rangle, \quad \mathbf{b} = \langle -3, 1 \rangle$

2. $\mathbf{a} = \langle -2, -8 \rangle, \quad \mathbf{b} = \langle 6, -4 \rangle$

3. $\mathbf{a} = \langle 4, 7, -1 \rangle, \quad \mathbf{b} = \langle -2, 1, 4 \rangle$

4. $\mathbf{a} = \langle -1, -2, -3 \rangle, \quad \mathbf{b} = \langle 2, 8, -6 \rangle$

5. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$

6. $\mathbf{a} = \mathbf{i} - \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j}$

7. $|\mathbf{a}| = 2, \quad |\mathbf{b}| = 3,$ the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/3$

8–13 ▪ Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

8. $\mathbf{a} = \langle 1, 2, 2 \rangle, \quad \mathbf{b} = \langle 3, 4, 0 \rangle$

9. $\mathbf{a} = \langle 6, 0, 2 \rangle, \quad \mathbf{b} = \langle 5, 3, -2 \rangle$

10. $\mathbf{a} = \langle 1, 2 \rangle, \quad \mathbf{b} = \langle 12, -5 \rangle$

11. $\mathbf{a} = \langle 3, 1 \rangle, \quad \mathbf{b} = \langle 2, 4 \rangle$

12. $\mathbf{a} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

13. $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{b} = 2\mathbf{j} - 3\mathbf{k}$

14–15 ▪ Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

14. $A(1, 2, 3), \quad B(6, 1, 5), \quad C(-1, -2, 0)$

15. $P(0, -1, 6), \quad Q(2, 1, -3), \quad R(5, 4, 2)$

16–21 ▪ Determine whether the given vectors are orthogonal, parallel, or neither.

16. $\mathbf{a} = \langle 2, -4 \rangle, \quad \mathbf{b} = \langle -1, 2 \rangle$

17. $\mathbf{a} = \langle 2, -4 \rangle, \quad \mathbf{b} = \langle 4, 2 \rangle$

18. $\mathbf{a} = \langle 2, 8, -3 \rangle, \quad \mathbf{b} = \langle -1, 2, 5 \rangle$

19. $\mathbf{a} = \langle -1, 5, 2 \rangle, \quad \mathbf{b} = \langle 4, 2, -3 \rangle$

20. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

21. $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

22. Find the values of x such that the vectors $\langle -3x, 2x \rangle$ and $\langle 4, x \rangle$ are orthogonal.

23. For what values of c is the angle between the vectors $\langle 1, 2, 1 \rangle$ and $\langle 1, 0, c \rangle$ equal to $60°$?

24–25 ▪ Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

24. $\mathbf{a} = \langle 2, 3 \rangle, \quad \mathbf{b} = \langle 4, 1 \rangle$

25. $\mathbf{a} = \langle 3, -1 \rangle, \quad \mathbf{b} = \langle 2, 3 \rangle$

10.3 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. -1
2. 20
3. -5
4. 0
5. -11
6. 1
7. 3
8. $\cos^{-1}\left(\frac{11}{15}\right) \approx 43^\circ$
9. $\cos^{-1}\left(\frac{13}{2\sqrt{95}}\right) \approx 48^\circ$
10. $\cos^{-1}\left(\frac{2}{13\sqrt{5}}\right) \approx 86^\circ$
11. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$
12. $\cos^{-1}\left(\frac{1}{7\sqrt{3}}\right) \approx 85^\circ$
13. $\cos^{-1}\left(-\frac{4}{\sqrt{78}}\right) \approx 117^\circ$
14. $114^\circ, 33^\circ, 33^\circ$
15. $43^\circ, 58^\circ, 79^\circ$
16. Parallel
17. Orthogonal
18. Neither
19. Orthogonal
20. Orthogonal
21. Orthogonal
22. $0, 6$
23. $2 \pm \sqrt{3}$
24. $\frac{11}{\sqrt{13}}, \left\langle \frac{22}{13}, \frac{33}{13} \right\rangle$
25. $\frac{3}{\sqrt{10}}, \left\langle \frac{9}{10}, -\frac{3}{10} \right\rangle$

10.3 SOLUTIONS

E Click here for exercises.

1. $\mathbf{a}\cdot\mathbf{b} = (2)(-3) + (5)(1) = -1$

2. $\mathbf{a}\cdot\mathbf{b} = (-2)(6) + (-8)(-4) = 20$

3. $\mathbf{a}\cdot\mathbf{b} = (4)(-2) + (7)(1) + (-1)(4) = -5$

4. $\mathbf{a}\cdot\mathbf{b} = (-1)(2) + (-2)(8) + (-3)(-6) = 0$

5. $\mathbf{a}\cdot\mathbf{b} = (2)(1) + (3)(-3) + (-4)(1) = -11$

6. $\mathbf{a}\cdot\mathbf{b} = (1)(1) + (0)(2) + (-1)(0) = 1$

7. $\mathbf{a}\cdot\mathbf{b} = (2)(3)\cos\frac{\pi}{3} = 6\cdot\frac{1}{2} = 3$

8. $|\mathbf{a}| = \sqrt{1^2+2^2+2^2} = 3$, $|\mathbf{b}| = \sqrt{3^2+4^2+0^2} = 5$, $\mathbf{a}\cdot\mathbf{b} = 3+8+0 = 11$, $\cos\theta = \frac{11}{3\cdot 5}$, so $\theta = \cos^{-1}\left(\frac{11}{15}\right) \approx 43°$.

9. $|\mathbf{a}| = \sqrt{6^2+0^2+2^2} = 2\sqrt{10}$, $|\mathbf{b}| = \sqrt{5^2+3^2+(-2)^2} = \sqrt{38}$, $\mathbf{a}\cdot\mathbf{b} = 30+0+(-4) = 26$, $\cos\theta = \frac{26}{2\sqrt{10}\sqrt{38}}$, so $\theta = \cos^{-1}\left(\frac{13}{2\sqrt{95}}\right) \approx 48°$.

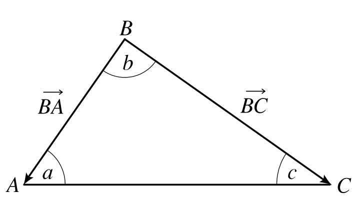
10. $|\mathbf{a}| = \sqrt{1^2+2^2} = \sqrt{5}$, $|\mathbf{b}| = \sqrt{12^2+(-5)^2} = \sqrt{13}$, $\mathbf{a}\cdot\mathbf{b} = 12-10 = 2$, $\cos\theta = \frac{2}{13\sqrt{5}}$, so $\theta = \cos^{-1}\left(\frac{2}{13\sqrt{5}}\right) \approx 86°$.

11. $|\mathbf{a}| = \sqrt{3^2+1^2} = \sqrt{10}$, $|\mathbf{b}| = \sqrt{2^2+4^2} = \sqrt{5}$, $\mathbf{a}\cdot\mathbf{b} = 6+4 = 10$, $\cos\theta = \frac{10}{\sqrt{10}\cdot 2\sqrt{5}} = \frac{\sqrt{2}}{2}$ and $\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$.

12. $|\mathbf{a}| = \sqrt{36+4+9} = 7$, $|\mathbf{b}| = \sqrt{3}$, $\mathbf{a}\cdot\mathbf{b} = 6-2-3 = 1$, $\cos\theta = \frac{1}{7\sqrt{3}}$, so $\theta = \cos^{-1}\left(\frac{1}{7\sqrt{3}}\right) \approx 85°$.

13. $|\mathbf{a}| = \sqrt{1+1+4} = \sqrt{6}$, $|\mathbf{b}| = \sqrt{4+9} = \sqrt{13}$, $\mathbf{a}\cdot\mathbf{b} = 0+2-6 = -4$, $\cos\theta = -\frac{4}{\sqrt{78}}$, so $\theta = \cos^{-1}\left(-\frac{4}{\sqrt{78}}\right) \approx 117°$.

14.



Let a, b and c be the angles at vertices A, B and C respectively. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

Thus

$$\cos a = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{1}{\sqrt{30\cdot 29}}\langle 5,-1,2\rangle\cdot\langle -2,-4,-3\rangle = \frac{1}{\sqrt{870}}(-10+4-6) = -\frac{12}{\sqrt{870}}$$

and $a = \cos^{-1}\left(-\frac{12}{\sqrt{870}}\right) \approx 114°$. Similarly

$$\cos b = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{1}{\sqrt{30\cdot 83}}\langle 5,-1,2\rangle\cdot\langle -7,-3,-5\rangle = \frac{1}{\sqrt{2490}}(35-3+10) = \frac{42}{\sqrt{2490}}$$

so $b = \cos^{-1}\left(\frac{42}{\sqrt{2490}}\right) \approx 33°$, and

$$\cos c = \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|} = \frac{1}{\sqrt{29\cdot 83}}\langle 2,4,3\rangle\cdot\langle 7,3,5\rangle = \frac{1}{\sqrt{2407}}(14+12+15) = \frac{41}{\sqrt{2407}}$$

so $c = \cos^{-1}\left(\frac{41}{\sqrt{2407}}\right) \approx 33°$.

Alternate Solution: Apply the Law of Cosines three times as follows: $\cos a = \frac{|\overrightarrow{BC}|^2 - |\overrightarrow{AB}|^2 - |\overrightarrow{AC}|^2}{2|\overrightarrow{AB}||\overrightarrow{AC}|}$,

$\cos b = \frac{|\overrightarrow{AC}|^2 - |\overrightarrow{AB}|^2 - |\overrightarrow{BC}|^2}{2|\overrightarrow{AB}||\overrightarrow{BC}|}$, and

$\cos c = \frac{|\overrightarrow{AB}|^2 - |\overrightarrow{AC}|^2 - |\overrightarrow{BC}|^2}{2|\overrightarrow{AC}||\overrightarrow{BC}|}$.

15. As in Problem 14, let p, q and r be the angles at vertices P, Q and R. Then p is the angle between vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, q is the angle between vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$, and r is the angle between vectors $\overrightarrow{RP}$ and $\overrightarrow{RQ}$. Thus

$\cos p = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle 2,2,-9\rangle\cdot\langle 5,5,-4\rangle}{\sqrt{89}\sqrt{66}} = \frac{56}{\sqrt{5874}}$, so

$p = \cos^{-1}\left(\frac{56}{\sqrt{5874}}\right) \approx 43°$;

$\cos q = \frac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|} = \frac{\langle -2,-2,9\rangle\cdot\langle 3,3,5\rangle}{\sqrt{89}\sqrt{43}} = \frac{33}{\sqrt{3827}}$,

so $q = \cos^{-1}\left(\frac{33}{\sqrt{3827}}\right) \approx 58°$; and

$\cos r = \frac{\overrightarrow{RP}\cdot\overrightarrow{RQ}}{|\overrightarrow{RP}||\overrightarrow{RQ}|} = \frac{\langle -5,-5,4\rangle\cdot\langle -3,-3,-5\rangle}{\sqrt{66}\sqrt{43}} = \frac{10}{\sqrt{2838}}$, so

$r = \cos^{-1}\left(\frac{10}{\sqrt{2838}}\right) \approx 79°$.

Alternate Solution: Apply the Law of Cosines three times as follows: $\cos p = \frac{|\overrightarrow{QR}|^2 - |\overrightarrow{PQ}|^2 - |\overrightarrow{PR}|^2}{2|\overrightarrow{PQ}||\overrightarrow{PR}|}$,

$\cos q = \frac{|\overrightarrow{PR}|^2 - |\overrightarrow{PQ}|^2 - |\overrightarrow{QR}|^2}{2|\overrightarrow{PQ}||\overrightarrow{QR}|}$, and

$\cos r = \frac{|\overrightarrow{PQ}|^2 - |\overrightarrow{PR}|^2 - |\overrightarrow{QR}|^2}{2|\overrightarrow{PR}||\overrightarrow{QR}|}$.

16. Since $\mathbf{a} = -2\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors (and thus not orthogonal).

17. $\mathbf{a}\cdot\mathbf{b} = 8 + (-8) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

18. $\mathbf{a}\cdot\mathbf{b} = -2 + 16 + (-15) \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

19. $\mathbf{a}\cdot\mathbf{b} = -4 + 10 + (-6) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

20. $\mathbf{a}\cdot\mathbf{b} = 3 + (-1) + (-2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

21. $\mathbf{a}\cdot\mathbf{b} = (-1)(3) + (2)(4) + (5)(-1) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

22. For the two vectors to be orthogonal, we need
$\langle -3x, 2x\rangle \cdot \langle 4, x\rangle = 0 \quad\Leftrightarrow\quad (-3x)(4) + (2x)(x) = 0$
$\Leftrightarrow\quad -12x + 2x^2 = 0 \quad\Leftrightarrow\quad 2x(x-6) = 0 \quad\Leftrightarrow\quad x = 0$
or $x = 6$.

23. Using Theorem 3, we need
$\langle 1,2,1\rangle \cdot \langle 1,0,c\rangle = |\langle 1,2,1\rangle|\,|\langle 1,0,c\rangle| \cos 60^\circ \quad\Leftrightarrow$
$1 + c = \sqrt{6}\sqrt{1+c^2}\cdot\frac{1}{2} \quad\Leftrightarrow\quad 2(1+c) = \sqrt{6}\sqrt{1+c^2}$.
Squaring both sides gives $6\left(1+c^2\right) = 4\left(1+2c+c^2\right)$.
Thus $6 + 6c^2 = 4 + 8c + 4c^2$ or $2c^2 - 8c + 2 = 0$ and
$c = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. Each of these values for c can be checked to show it gives a solution.

24. $|\mathbf{a}| = \sqrt{4+9} = \sqrt{13}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{2\cdot 4 + 3\cdot 1}{\sqrt{13}} = \frac{11}{\sqrt{13}}.$$

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \frac{11}{\sqrt{13}}\frac{\mathbf{a}}{|\mathbf{a}|} = \tfrac{11}{\sqrt{13}}\cdot\tfrac{1}{\sqrt{13}}\langle 2,3\rangle = \tfrac{11}{13}\langle 2,3\rangle$$
$$= \left\langle \tfrac{22}{13}, \tfrac{33}{13}\right\rangle$$

25. $|\mathbf{a}| = \sqrt{9+1} = \sqrt{10}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{3\cdot 2 - 1\cdot 3}{\sqrt{10}} = \frac{3}{\sqrt{10}}.$$

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \frac{3}{\sqrt{10}}\frac{\mathbf{a}}{|\mathbf{a}|} = \tfrac{3}{\sqrt{10}}\cdot\tfrac{1}{\sqrt{10}}\langle 3,-1\rangle = \tfrac{3}{10}\langle 3,-1\rangle$$
$$= \left\langle \tfrac{9}{10}, -\tfrac{3}{10}\right\rangle$$