10.8 ARC LENGTH AND CURVATURE

A Click here for answers.

S Click here for solutions.

1–3 ▪ Find the length of the given curve.

1. $\mathbf{r}(t) = \langle 2t, 3\sin t, 3\cot t \rangle, \quad a \leqslant t \leqslant b$

2. $\mathbf{r}(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle, \quad 0 \leqslant t \leqslant 2\pi$

3. $\mathbf{r}(t) = 6t\,\mathbf{i} + 3\sqrt{2}t^2\,\mathbf{j} + 2t^3\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

4. $x = 2t,\ y = t^2,\ z = t^2, \quad 0 \leqslant t \leqslant 1$

5. $x = t\cos t,\ y = t\sin t,\ z = t, \quad 0 \leqslant t \leqslant \pi/2$

6–8 ▪ Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

6. $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j}$

7. $\mathbf{r}(t) = (1 + 2t)\,\mathbf{i} + (3 + t)\,\mathbf{j} - 5t\,\mathbf{k}$

8. $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j} + \cos 2t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi/2$

9–14 ▪

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.
(b) Use Formula 9 to find the curvature.

9. $\mathbf{r}(t) = \langle \sin 4t, 3t, \cos 4t \rangle$

10. $\mathbf{r}(t) = \langle 6t, 3\sqrt{2}t^2, 2t^3 \rangle$

11. $\mathbf{r}(t) = \langle \sqrt{2}\cos t, \sin t, \sin t \rangle$

12. $\mathbf{r}(t) = \langle \frac{1}{3}t^3, t^2, 2t \rangle$

13. $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$

14. $\mathbf{r}(t) = \langle t^2, 2t^3/3, t \rangle$

15–19 ▪ Use Theorem 10 to find the curvature.

15. $\mathbf{r}(t) = \mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$

16. $\mathbf{r}(t) = (1 + t)\mathbf{i} + (1 - t)\mathbf{j} + 3t^2\,\mathbf{k}$

17. $\mathbf{r}(t) = 2t^3\,\mathbf{i} - 3t^2\,\mathbf{j} + 6t\,\mathbf{k}$

18. $\mathbf{r}(t) = (t^2 + 2)\mathbf{i} + (t^2 - 4t)\mathbf{j} + 2t\,\mathbf{k}$

19. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$

20. Find the curvature of $\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ at the point (0, 1, 1).

21–23 ▪ Use Formula 11 to find the curvature.

21. $y = \sqrt{x}$

22. $y = \sin x$

23. $y = \ln x$

24–25 ▪ Use the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t (see Exercise 32 in the text) to find the curvature of the parametric curve.

24. $x = t^3, \quad y = t^2$

25. $x = t\sin t, \quad y = t\cos t$

10.8 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\sqrt{13}\,(b-a)$

2. $\sqrt{3}\,(e^{2\pi}-1)$

3. 8

4. $\sqrt{3}+\frac{\sqrt{2}}{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$

5. $\ln\left(\sqrt{\frac{\pi^2}{4}+2}+\frac{\pi}{2}\right)+\frac{\pi}{4}\sqrt{\frac{\pi^2}{4}+2}-\ln\sqrt{2}$

6. $\mathbf{r}\,(t\,(s))=\left(\frac{1}{\sqrt{2}}s+1\right)\left[\sin\left(\ln\left(\frac{1}{\sqrt{2}}s+1\right)\right)\mathbf{i}+\cos\left(\ln\left(\frac{1}{\sqrt{2}}s+1\right)\right)\mathbf{j}\right]$

7. $\mathbf{r}\,(t\,(s))=\left(1+\frac{2}{\sqrt{30}}s\right)\mathbf{i}+\left(3+\frac{1}{\sqrt{30}}s\right)\mathbf{j}-\frac{5}{\sqrt{30}}s\,\mathbf{k}$

8. $\mathbf{r}\,(t\,(s))=\cos^3\left[\frac{1}{2}\cos^{-1}\left(1-\frac{4}{5}s\right)\right]\mathbf{i}+\sin^3\left[\frac{1}{2}\cos^{-1}\left(1-\frac{4}{5}s\right)\right]\mathbf{j}+\left[1-\frac{4}{5}s\right]\mathbf{k}$

9. (a) $\frac{1}{5}\langle 4\cos 4t, 3, -4\sin 4t\rangle$, $\langle -\sin 4t, 0, -\cos 4t\rangle$
   (b) $\frac{16}{25}$

10. (a) $\dfrac{1}{1+t^2}\langle 1,\sqrt{2}t,t^2\rangle$, $\dfrac{1}{\sqrt{2}\,(1+t^2)}\langle -2t,\sqrt{2}\,(1-t^2),2t\rangle$
   (b) $\dfrac{1}{3\sqrt{2}\,(1+t^2)^2}$

11. (a) $\frac{1}{\sqrt{2}}\langle -\sqrt{2}\sin t,\cos t,\cos t\rangle$, $\frac{1}{\sqrt{2}}\langle -\sqrt{2}\cos t,-\sin t,-\sin t\rangle$
   (b) $\frac{1}{\sqrt{2}}$

12. (a) $\dfrac{1}{t^2+2}\langle t^2,2t,2\rangle$, $\dfrac{1}{t^2+2}\langle 2t,2-t^2,-2t\rangle$
   (b) $\dfrac{2}{(t^2+2)^2}$

13. (a) $\dfrac{1}{e^{2t}+1}\langle\sqrt{2}e^{2t},e^{2t},-1\rangle$, $\dfrac{1}{e^{2t}+1}\langle 1-e^{2t},\sqrt{2}e^t,\sqrt{2}e^t\rangle$
   (b) $\dfrac{\sqrt{2}e^{2t}}{(e^{2t}+1)^2}$

14. (a) $\dfrac{1}{2t^2+1}\langle 2t,2t^2,1\rangle$, $\dfrac{1}{2t^2+1}\langle 1-2t^2,2t,-2t\rangle$
   (b) $\dfrac{2}{(2t^2+1)^2}$

15. $\dfrac{2}{(4t^2+1)^{3/2}}$

16. $\dfrac{3}{(1+18t^2)^{3/2}}$

17. $\dfrac{\sqrt{1+4t^2+t^4}}{6\,(t^4+t^2+1)^{3/2}}$

18. $\dfrac{\sqrt{6}}{2\,(2t^2-4t+5)^{3/2}}$

19. $\dfrac{\sqrt{2}}{(1+\cos^2 t)^{3/2}}$

20. $\frac{\sqrt{2}}{4}$

21. $\dfrac{2}{(4x+1)^{3/2}}$

22. $\dfrac{|\sin x|}{(1+\cos^2 x)^{3/2}}$

23. $\dfrac{|x|}{(x^2+1)^{3/2}}$

24. $\dfrac{6}{|t|\,(9t^2+4)^{3/2}}$

25. $\dfrac{2+t^2}{(1+t^2)^{3/2}}$

10.8 SOLUTIONS

E Click here for exercises.

1. $\mathbf{r}'(t) = \langle 2, 3\cos t, -3\sin t\rangle$,

$|\mathbf{r}'(t)| = \sqrt{4 + 9\cos^2 t + 9\sin^2 t} = \sqrt{13}$,

$L = \int_a^b \sqrt{13}\,dt = \sqrt{13}\,(b-a)$

2. $\mathbf{r}'(t) = \langle e^t, e^t(\sin t + \cos t), e^t(\cos t - \sin t)\rangle$,

$|\mathbf{r}'(t)| = e^t\sqrt{1 + 2\sin^2 t + 2\cos^2 t} = \sqrt{3}e^t$,

$L = \int_0^{2\pi} \sqrt{3}e^t\,dt = \sqrt{3}\left(e^{2\pi} - 1\right)$

3. $\mathbf{r}'(t) = \langle 6, 6\sqrt{2}t, 6t^2\rangle$,

$|\mathbf{r}'(t)| = 6\sqrt{1 + 2t^2 + t^4} = 6\left(1 + t^2\right)$,

$L = \int_0^1 6\left(1 + t^2\right)\,dt = \left[\frac{1}{3}\cdot 6\left(t + t^3\right)\right]_0^1 = \frac{24}{3} = 8$

4. $\mathbf{r}'(t) = \langle 2, 2t, 2t\rangle$, $|\mathbf{r}'(t)| = 2\sqrt{1 + 2t^2}$

$L = \int_0^1 \sqrt{1 + 2t^2}\,dt = \int_a^b \sqrt{2}\sec^3\theta\,d\theta$

$= \frac{\sqrt{2}}{2}\left[\ln|\sec\theta + \tan\theta| + \tan\theta\sec\theta\right]_a^b$

$= \frac{\sqrt{2}}{2}\left[\ln\left|\sqrt{1+2t^2} + \sqrt{2}t\right| + \sqrt{2}t\sqrt{1+2t^2}\right]_0^1$

(or use Formula 21)

$= \frac{\sqrt{2}}{2}\left[\ln\left|\sqrt{3} + \sqrt{2}\right| + \sqrt{2}\sqrt{3}\right]$

$= \sqrt{3} + \frac{\sqrt{2}}{2}\ln\left(\sqrt{2} + \sqrt{3}\right)$

5. $\mathbf{r}'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 1\rangle$,

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + t^2\sin^2 t + \sin^2 t + t\cos^2 t + 1}$

$= \sqrt{t^2 + 2}$

$L = \int_0^{\pi/2} \sqrt{t^2 + 2}\,dt = \int_a^b \sec^3\theta\,d\theta$

$= \left[\ln\left|\sqrt{t^2+2} + t\right| + \frac{1}{2}t\sqrt{t^2+2}\right]_0^{\pi/2}$

(or use Formula 21)

$= \ln\left(\sqrt{\frac{\pi^2}{4} + 2} + \frac{\pi}{2}\right) + \frac{\pi}{4}\sqrt{\frac{\pi^2}{4} + 2} - \ln\sqrt{2}$

6. $\mathbf{r}'(t) = e^t(\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j}$,

$ds/dt = |\mathbf{r}'(t)| = e^t\sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2}$

$= e^t\sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}e^t$

$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}e^u\,du = \sqrt{2}\left(e^t - 1\right) \Rightarrow$

$\frac{1}{\sqrt{2}}s + 1 = e^t \Rightarrow t(s) = \ln\left(\frac{1}{\sqrt{2}}s + 1\right)$. Therefore,

$\mathbf{r}(t(s)) = \left(\frac{1}{\sqrt{2}}s + 1\right)\left[\sin\left(\ln\left(\frac{1}{\sqrt{2}}s + 1\right)\right)\mathbf{i} + \cos\left(\ln\left(\frac{1}{\sqrt{2}}s + 1\right)\right)\mathbf{j}\right]$

7. $\mathbf{r}'(t) = 2\,\mathbf{i} + \mathbf{j} - 5\,\mathbf{k}$,

$ds/dt = |\mathbf{r}'(t)| = \sqrt{4 + 1 + 25} = \sqrt{30}$ and

$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{30}\,du = \sqrt{30}\,t$

$\Rightarrow t(s) = \frac{1}{\sqrt{30}}s$. Therefore,

$\mathbf{r}(t(s)) = \left(1 + \frac{2}{\sqrt{30}}s\right)\mathbf{i} + \left(3 + \frac{1}{\sqrt{30}}s\right)\mathbf{j} - \frac{5}{\sqrt{30}}s\,\mathbf{k}$.

8. $|\mathbf{r}'(t)|$

$= \sqrt{\left(-3\cos^2 t\sin t\right)^2 + \left(3\sin^2 t\cos t\right)^2 + (-2\sin 2t)^2}$

$= \sqrt{9\sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right) + 4\sin^2 2t}$

$= \sqrt{\frac{9}{4}(2\sin t\cos t)^2 + 4\sin^2 2t} = \sqrt{\left(\frac{9}{4} + 4\right)\sin^2 2t}$

$= \frac{5}{2}\sin 2t$

Then for $0 \le t \le \frac{\pi}{2}$,

$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \frac{5}{2}\int_0^t \sin 2u\,du$

$= -\frac{5}{4}\left[\cos 2u\right]_0^t = \frac{5}{4}(1 - \cos 2t)$

Therefore, $t(s) = \frac{1}{2}\cos^{-1}\left(1 - \frac{4}{5}s\right)$ and

$\mathbf{r}(t(s)) = \cos^3\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{4}{5}s\right)\right]\mathbf{i} + \sin^3\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{4}{5}s\right)\right]\mathbf{j} + \cos\left[2\cdot\frac{1}{2}\cos^{-1}\left(1 - \frac{4}{5}s\right)\right]\mathbf{k}$

$= \cos^3\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{4}{5}s\right)\right]\mathbf{i} + \sin^3\left[\frac{1}{2}\cos^{-1}\left(1 - \frac{4}{5}s\right)\right]\mathbf{j} + \left[1 - \frac{4}{5}s\right]\mathbf{k}$

9. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{16+9}}\langle 4\cos 4t, 3, -4\sin 4t\rangle$

$= \frac{1}{5}\langle 4\cos 4t, 3, -4\sin 4t\rangle$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{5}{16\cdot 5}\langle -16\sin 4t, 0, -16\cos 4t\rangle$

$= \langle -\sin 4t, 0, -\cos 4t\rangle$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{16}{5\cdot 5} = \dfrac{16}{25}$

10. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{6\left(1+t^2\right)}\left\langle 6, 6\sqrt{2}t, 6t^2\right\rangle$

$= \dfrac{1}{1+t^2}\left\langle 1, \sqrt{2}t, t^2\right\rangle$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

$= \dfrac{1}{|\mathbf{T}'(t)|}\left[-\dfrac{2t}{\left(1+t^2\right)^2}\left\langle 1, \sqrt{2}t, t^2\right\rangle + \dfrac{1}{1+t^2}\left\langle 0, \sqrt{2}, 2t\right\rangle\right]$

$= \dfrac{1+t^2}{\sqrt{2}}\left[\dfrac{1}{\left(1+t^2\right)^2}\left\langle -2t, \sqrt{2}\left(1-t^2\right), 2t\right\rangle\right]$

$= \dfrac{1}{\sqrt{2}\left(1+t^2\right)}\left\langle -2t, \sqrt{2}\left(1-t^2\right), 2t\right\rangle$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{2}}{1+t^2}\cdot\dfrac{1}{6\left(1+t^2\right)}$

$= \dfrac{1}{3\sqrt{2}\left(1+t^2\right)^2}$

11. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

$$= \frac{1}{\sqrt{2\sin^2 t + 2\cos^2 t}} \left\langle -\sqrt{2}\sin t, \cos t, \cos t \right\rangle$$

$$= \tfrac{1}{\sqrt{2}} \left\langle -\sqrt{2}\sin t, \cos t, \cos t \right\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

$$= \frac{1}{\sqrt{2\cos^2 t + 2\sin^2 t}} \left\langle -\sqrt{2}\cos t, -\sin t, -\sin t \right\rangle$$

$$= \tfrac{1}{\sqrt{2}} \left\langle -\sqrt{2}\cos t, -\sin t, -\sin t \right\rangle$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2}}$

12. (a) $\mathbf{r}'(t) = \left\langle t^2, 2t, 2 \right\rangle \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{t^2+2}\left\langle t^2, 2t, 2 \right\rangle.$$

$$\mathbf{T}'(t) = \frac{-2t}{(t^2+2)^2}\left\langle t^2, 2t, 2 \right\rangle + \frac{1}{t^2+2}\left\langle 2t, 2, 0 \right\rangle$$

(by Theorem 10.7.5 #3)

$$= \frac{1}{(t^2+2)^2}\left\langle -2t^3, -4t^2, -4t \right\rangle + \frac{1}{(t^2+2)^2}\left\langle 2t^3 + 4t, 2t^2 + 4, 0 \right\rangle$$

$$= \frac{1}{(t^2+2)^2}\left\langle 4t, 4 - 2t^2, -4t \right\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(t^2+2)^2}\sqrt{16t^2 + (16 - 16t^2 + 4t^4) + 16t^2}$$

$$= \frac{1}{(t^2+2)^2}\sqrt{4t^4 + 16t^2 + 16}$$

$$= \frac{1}{(t^2+2)^2}\sqrt{4(t^2+2)^2} = \frac{2(t^2+2)}{(t^2+2)^2}$$

$$= \frac{2}{t^2+2}$$

Thus

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1/(t^2+2)^2}{2/(t^2+2)}\left\langle 4t, 4 - 2t^2, -4t \right\rangle$$

$$= \frac{1}{t^2+2}\left\langle 2t, 2 - t^2, -2t \right\rangle$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/(t^2+2)}{t^2+2} = \dfrac{2}{(t^2+2)^2}$

13. (a) $\mathbf{T}(t) = \dfrac{1}{\sqrt{2 + e^{2t} + e^{-2t}}}\left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle$

$$= \frac{1}{e^t + e^t}\left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle$$

$$= \frac{1}{e^{2t}+1}\left\langle \sqrt{2}e^{2t}, e^{2t}, -1 \right\rangle$$

$$\mathbf{T}'(t) = \frac{-2e^{2t}}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^t, e^{2t}, -1 \right\rangle + \frac{1}{e^{2t}+1}\left\langle \sqrt{2}e^t, 2e^{2t}, 0 \right\rangle$$

$$= \frac{1}{(e^{2t}+1)^2}\left\langle -2\sqrt{2}e^{3t} + \sqrt{2}e^{3t} + \sqrt{2}e^t, -2e^{4t} + 2e^{4t} + 2e^{2t}, 2e^{2t} \right\rangle$$

$$= \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\left(e^t - e^{3t}\right), 2e^{2t}, 2e^{2t} \right\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(e^{2t}+1)^2}\sqrt{2\left(e^{2t} + e^{6t} - 2e^{4t}\right) + 8e^{4t}}$$

$$= \frac{\sqrt{2}e^t}{(e^{2t}+1)^2}\left(e^{2t}+1\right) = \frac{\sqrt{2}e^t}{e^{2t}+1}$$

$$\mathbf{N}(t) = \frac{1}{(e^{2t}+1)^2}\,\frac{e^{2t}+1}{\sqrt{2}e^t}\left\langle \sqrt{2}\left(e^t - e^{3t}\right), 2e^{2t}, 2e^{2t} \right\rangle$$

$$= \frac{1}{\sqrt{2}\left(e^{2t}+1\right)e^t}\left\langle \sqrt{2}\left(e^t - e^{3t}\right), 2e^{2t}, 2e^{2t} \right\rangle$$

$$= \frac{1}{e^{2t}+1}\left\langle 1 - e^{2t}, \sqrt{2}e^t, \sqrt{2}e^t \right\rangle$$

(b) $\kappa(t) = \dfrac{\sqrt{2}e^t}{(e^{2t}+1)} \cdot \dfrac{1}{e^t + e^{-t}} = \dfrac{\sqrt{2}e^{2t}}{(e^{2t}+1)^2}$

14. (a) $\mathbf{T}(t) = \dfrac{1}{\sqrt{4t^2 + 4t^4 + 1}}\left\langle 2t, 2t^2, 1 \right\rangle$

$$= \frac{1}{2t^2+1}\left\langle 2t, 2t^2, 1 \right\rangle$$

$$\mathbf{T}'(t) = -\left(2t^2+1\right)^{-2}(4t)\left\langle 2t, 2t^2, 1 \right\rangle + \left(2t^2+1\right)^{-1}\left\langle 2, 4t, 0 \right\rangle$$

$$= \frac{1}{(2t^2+1)^2}\left\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4, -4t \right\rangle$$

$$= \frac{1}{(2t^2+1)^2}\left\langle 1 - 2t^2, 2t, -2t \right\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(2t^2+1)^2}\sqrt{1 - 4t^2 + 4t^4 + 8t^2} = \frac{2}{2t^2+1}$$

$$\mathbf{N}(t) = \frac{2}{(2t^2+1)^2} \cdot \frac{2t^2+1}{2}\left\langle 1 - 2t^2, 2t, -2t \right\rangle$$

$$= \frac{1}{2t^2+1}\left\langle 1 - 2t^2, 2t, -2t \right\rangle$$

(b) $\kappa(t) = \dfrac{2}{2t^2+1} \cdot \dfrac{1}{2t^2+1} = \dfrac{2}{(2t^2+1)^2}$

15. $\mathbf{r}'(t) = \mathbf{j} - 2t\,\mathbf{k}$, $\mathbf{r}''(t) = -2\,\mathbf{k}$, $|\mathbf{r}'(t)|^3 = \left(4t^2+1\right)^{3/2}$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |-2\mathbf{i}| = 2$,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2}{(4t^2+1)^{3/2}}$$

16. $\mathbf{r}'(t) = \langle 1, -1, 6t \rangle$, $\mathbf{r}''(t) = \langle 0, 0, 6 \rangle$,

$|\mathbf{r}'(t)|^3 = \left(\sqrt{2 + 36t^2}\right)^3 = \left[2\left(1 + 18t^2\right)\right]^{3/2}$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle -6, -6, 0 \rangle| = 6\sqrt{2}$,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{6\sqrt{2}}{[2(1+18t^2)]^{3/2}} = \frac{3}{(1+18t^2)^{3/2}}$$

17. $\mathbf{r}'(t) = \langle 6t^2, -6t, 6\rangle$, $\mathbf{r}''(t) = \langle 12t, -6, 0\rangle$,

$$|\mathbf{r}'(t)|^3 = 6^3\left(t^4+t^2+1\right)^{3/2},$$

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \left|36\left\langle 1, 2t, t^2\right\rangle\right| = 36\sqrt{1+4t^2+t^4},$$

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{36\sqrt{1+4t^2+t^4}}{6^3\left(t^4+t^2+1\right)^{3/2}} = \frac{\sqrt{1+4t^2+t^4}}{6\left(t^4+t^2+1\right)^{3/2}}$$

18. $\mathbf{r}'(t) = \langle 2t, 2t-4, 2\rangle$, $\mathbf{r}''(t) = \langle 2, 2, 0\rangle$,

$$|\mathbf{r}'(t)|^3 = \left(4t^2+4t^2-16t+16+4\right)^{3/2} = 8\left(2t^2-4t+5\right)^{3/2},$$

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = 4\,|\langle -1, 1, 2\rangle| = 4\sqrt{6},$$

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{4\sqrt{6}}{8\left(2t^2-4t+5\right)^{3/2}} = \frac{\sqrt{6}}{2\left(2t^2-4t+5\right)^{3/2}}$$

19. $\mathbf{r}'(t) = \langle \cos t, -\sin t, \cos t\rangle$,

$$\mathbf{r}''(t) = \langle -\sin t, -\cos t, -\sin t\rangle,$$

$$|\mathbf{r}'(t)|^3 = \left(\sqrt{\cos^2 t+1}\right)^3,$$

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = |\langle 1, 0, -1\rangle| = \sqrt{2},$$

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{2}}{\left(1+\cos^2 t\right)^{3/2}}$$

20. $\mathbf{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle$. The point $(0, 1, 1)$ corresponds to $t = 0$, and $\mathbf{r}'(0) = \left\langle \sqrt{2}, 1, -1\right\rangle$ $\Rightarrow$ $|\mathbf{r}'(0)| = \sqrt{\left(\sqrt{2}\right)^2 + 1^2 + (-1)^2} = 2$.

$\mathbf{r}''(t) = \left\langle 0, e^t, e^{-t}\right\rangle \Rightarrow \mathbf{r}''(0) = \langle 0, 1, 1\rangle$.

$\mathbf{r}'(0)\times\mathbf{r}''(0) = \left\langle 2, -\sqrt{2}, \sqrt{2}\right\rangle$,

$$|\mathbf{r}'(0)\times\mathbf{r}''(0)| = \sqrt{2^2+\left(-\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2} = \sqrt{8} = 2\sqrt{2}$$

Then $\kappa(0) = \dfrac{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{2\sqrt{2}}{2^3} = \dfrac{\sqrt{2}}{4}$.

21. $y' = \dfrac{1}{2\sqrt{x}}$, $y'' = -\dfrac{1}{4(x)^{3/2}}$,

$$\kappa(x) = \frac{|y''(x)|}{\left[1+\left(y'(x)\right)^2\right]^{3/2}} = \frac{1}{4\left|x^{3/2}\right|}\frac{1}{\left[1+1/(4x)\right]^{3/2}} = \frac{2}{(4x+1)^{3/2}}$$

22. $y' = \cos x$, $y'' = -\sin x$,

$$\kappa(x) = \frac{|y''(x)|}{\left[1+\left(y'(x)\right)^2\right]^{3/2}} = \frac{|\sin x|}{\left(1+\cos^2 x\right)^{3/2}}$$

23. $y' = \dfrac{1}{x}$, $y'' = -\dfrac{1}{x^2}$,

$$\kappa(x) = \frac{|y''(x)|}{\left[1+\left(y'(x)\right)^2\right]^{3/2}} = \left|\frac{-1}{x^2}\right|\frac{1}{\left(1+1/x^2\right)^{3/2}} = \frac{1}{x^2}\frac{\left(x^2\right)^{3/2}}{\left(x^2+1\right)^{3/2}} = \frac{|x|}{\left(x^2+1\right)^{3/2}}$$

24.

$$\kappa(t) = \frac{|\dot{x}\ddot{y}-\ddot{x}\dot{y}|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}} = \frac{\left|\left(3t^2\right)(2)-(6t)(2t)\right|}{\left(9t^4+4t^2\right)^{3/2}} = \frac{6t^2}{\left(t^2\right)^{3/2}\left(9t^2+4\right)^{3/2}} = \frac{6t^2}{|t|^3\left(9t^2+4\right)^{3/2}} = \frac{6}{|t|\left(9t^2+4\right)^{3/2}}$$

25.

$$\kappa(t) = \frac{|\dot{x}\ddot{y}-\ddot{x}\dot{y}|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}} = \frac{\left|(\sin t+t\cos t)(-2\sin t-t\cos t) - (2\cos t-t\sin t)(\cos t-t\sin t)\right|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}$$

$$= \frac{\left|\left(-2\sin^2 t-3t\sin t\cos t-t^2\cos^2 t\right) - \left(2\cos^2 t-3t\cos t\sin t+t^2\sin^2 t\right)\right|}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}$$

$$= \frac{\left|-\left(\sin^2 t+\cos^2 t\right)\left(2+t^2\right)\right|}{\left(\sin^2 t+2t\cos t\sin t+\cos^2 t + t^2\cos^2 t-2t\sin t\cos t+t^2\sin t\right)^{3/2}} = \frac{2+t^2}{\left(1+t^2\right)^{3/2}}$$