

10.9

MOTION IN SPACE: VELOCITY AND ACCELERATION

A Click here for answers.

1. Find the velocity, acceleration, and speed of a particle with position function

$$\mathbf{r}(t) = \langle \sqrt{t}, 1 - t \rangle$$

Sketch the path of the particle and draw the velocity and acceleration vectors for $t = 1$.

2–7 ■ Find the velocity, acceleration, and speed of a particle with the given position function.

2. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$
 3. $\mathbf{r}(t) = \langle t^3, t^2 + 1, t^3 - 1 \rangle$
 4. $\mathbf{r}(t) = \langle \sqrt{t}, t, t\sqrt{t} \rangle$
 5. $\mathbf{r}(t) = (1/t)\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$

S Click here for solutions.

6. $\mathbf{r}(t) = e^t \mathbf{i} + 2t \mathbf{j} + e^{-t} \mathbf{k}$

7. $\mathbf{r}(t) = \cosh t \mathbf{i} + \sinh t \mathbf{j} + t \mathbf{k}$

8. A gun has muzzle speed 120 m/s. What angle of elevation should be used to hit an object 500 m away?

9–12 ■ Find the tangential and normal components of the acceleration vector.

9. $\mathbf{r}(t) = (t^2 + 4)\mathbf{i} + (2t - 3)\mathbf{j}$

10. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$

11. $\mathbf{r}(t) = t \mathbf{i} + 4 \sin t \mathbf{j} + 4 \cos t \mathbf{k}$

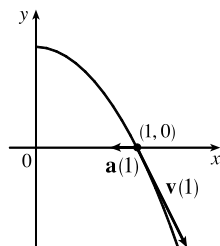
12. $\mathbf{r}(t) = t^3 \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k}$

10.9 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\left\langle \frac{1}{2}t^{-1/2}, -1 \right\rangle, \left\langle -\frac{1}{4}t^{-3/2}, 0 \right\rangle, \sqrt{\frac{1}{4}t^{-1} + 1}$



2. $\langle 1, 2t, 3t^2 \rangle, \langle 0, 2, 6t \rangle, \sqrt{1 + 4t^2 + 9t^4}$

3. $\langle 3t^2, 2t, 3t^2 \rangle, \langle 6t, 2, 6t \rangle, |t| \sqrt{18t^2 + 4}$

4. $\left\langle \frac{1}{2}t^{-1/2}, 1, \frac{3}{2}t^{1/2} \right\rangle, \left\langle -\frac{1}{4}t^{-3/2}, 0, \frac{3}{4}t^{-1/2} \right\rangle, \frac{1}{2} \sqrt{\frac{1 + 4t + 9t^2}{t}}$

5. $\langle -t^{-2}, 0, 2t \rangle, \langle 2t^{-3}, 0, 2 \rangle, \frac{1}{t^2} \sqrt{4t^6 + 1}$

6. $\langle e^t, 2, -e^{-t} \rangle, \langle e^t, 0, e^{-t} \rangle, \sqrt{e^{2t} + 4 + e^{-2t}}$

7. $\langle \sinh t, \cosh t, 1 \rangle, \langle \cosh t, \sinh t, 0 \rangle, \sqrt{\cosh 2t + 1}$

8. 9.9°

9. $\frac{2t}{\sqrt{t^2 + 1}}, \frac{2}{\sqrt{t^2 + 1}}$

10. $\frac{\sin t}{\sqrt{2(1 - \cos t)}}, \frac{\sqrt{1 - \cos t}}{\sqrt{2}}$

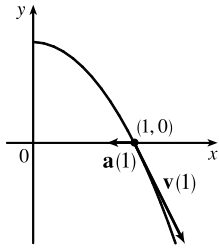
11. 0, 4

12. $\frac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2 + 1}}, \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{9t^4 + 4t^2 + 1}}$

10.9 SOLUTIONS

[Click here for exercises.](#)

1. $\mathbf{r}(t) = \langle \sqrt{t}, 1-t \rangle \Rightarrow \mathbf{v}(t) = \langle \frac{1}{2}t^{-1/2}, -1 \rangle$,
 $\mathbf{a}(t) = \langle -\frac{1}{4}t^{-3/2}, 0 \rangle$, $|\mathbf{v}(t)| = \sqrt{\frac{1}{4}t^{-1} + 1}$
 At $t = 1$: $\mathbf{v}(1) = \langle \frac{1}{2}, -1 \rangle$, $\mathbf{a}(1) = \langle -\frac{1}{4}, 0 \rangle$
 Since $x^2 = t$, $y = 1 - t = 1 - x^2$, but $x = \sqrt{t}$, so $x \geq 0$.



2. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$,
 $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2, 6t \rangle$,
 $|\mathbf{v}(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$
3. $\mathbf{r}(t) = \langle t^3, t^2 + 1, t^3 - 1 \rangle \Rightarrow \mathbf{v}(t) = \langle 3t^2, 2t, 3t^2 \rangle$,
 $\mathbf{a}(t) = \langle 6t, 2, 6t \rangle$,
 $|\mathbf{v}(t)| = \sqrt{9t^4 + 4t^2 + 9t^4} = \sqrt{18t^4 + 4t^2}$
 $= |t| \sqrt{18t^2 + 4}$
4. $\mathbf{r}(t) = \langle \sqrt{t}, t, t\sqrt{t} \rangle \Rightarrow \mathbf{v}(t) = \langle \frac{1}{2}t^{-1/2}, 1, \frac{3}{2}t^{1/2} \rangle$,
 $\mathbf{a}(t) = \langle -\frac{1}{4}t^{-3/2}, 0, \frac{3}{4}t^{-1/2} \rangle$,
 $|\mathbf{v}(t)| = \sqrt{\frac{1}{4}t^{-1} + 1 + \frac{9}{4}t} = \frac{1}{2} \sqrt{\frac{1 + 4t + 9t^2}{t}}$
5. $\mathbf{r}(t) = \langle 1/t, 1, t^2 \rangle \Rightarrow \mathbf{v}(t) = \langle -t^{-2}, 0, 2t \rangle$,
 $\mathbf{a}(t) = \langle 2t^{-3}, 0, 2 \rangle$, $|\mathbf{v}(t)| = \sqrt{t^{-4} + 4t^2} = \frac{1}{t^2} \sqrt{4t^6 + 1}$
6. $\mathbf{r}(t) = \langle e^t, 2t, e^{-t} \rangle \Rightarrow \mathbf{v}(t) = \langle e^t, 2, -e^{-t} \rangle$,
 $\mathbf{a}(t) = \langle e^t, 0, -e^{-t} \rangle$, $|\mathbf{v}(t)| = \sqrt{e^{2t} + 4 + e^{-2t}}$
7. $\mathbf{r}(t) = \langle \cosh t, \sinh t, t \rangle \Rightarrow$
 $\mathbf{v}(t) = \langle \sinh t, \cosh t, 1 \rangle$, $\mathbf{a}(t) = \langle \cosh t, \sinh t, 0 \rangle$,
 $|\mathbf{v}(t)| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{\cosh 2t + 1}$
 Recall that $\cosh^2 t - \sinh^2 t = 1$.

8. Here the initial speed $v_0 = 120$ m/s; let α be the angle of elevation. Assuming the object is lying flat on the ground, the object will be hit at time $t = \frac{240 \sin \alpha}{g}$ s (again refer to

Example 5). Then $\frac{(120)^2 \sin 2\alpha}{g} = 500$ or

$$\sin 2\alpha = \frac{500g}{(120)^2} = \frac{5g}{144} \text{ and } 2\alpha = \sin^{-1} \frac{5g}{144} \approx 19.9^\circ, \text{ so}$$

$$\alpha \approx 9.9^\circ.$$

9. $\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j}$, $|\mathbf{r}'(t)| = 2\sqrt{t^2 + 1}$, $\mathbf{r}''(t) = 2\mathbf{i}$. Thus

$$a_T = \frac{4t}{2\sqrt{t^2 + 1}} = \frac{2t}{\sqrt{t^2 + 1}} \text{ and}$$

$$a_N = \frac{|-4\mathbf{k}|}{2\sqrt{t^2 + 1}} = \frac{2}{\sqrt{t^2 + 1}}.$$

10. $\mathbf{r}'(t) = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$,
 $|\mathbf{r}'(t)| = \sqrt{1 - 2\cos t + 1} = \sqrt{2(1 - \cos t)}$,
 $\mathbf{r}''(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$. Thus $a_T = \frac{\sin t}{\sqrt{2(1 - \cos t)}}$ and
 $a_N = \frac{|(\cos t - \cos^2 t - \sin^2 t)\mathbf{k}|}{\sqrt{2(1 - \cos t)}} = \frac{\sqrt{[(\cos t) - 1]^2}}{\sqrt{2}\sqrt{1 - \cos t}}$
 $= \frac{1}{\sqrt{2}} \sqrt{\frac{(1 - \cos t)^2}{1 - \cos t}} = \frac{\sqrt{1 - \cos t}}{\sqrt{2}}$

11. $\mathbf{r}'(t) = \mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1 + 4} = \sqrt{5}$,
 $\mathbf{r}''(t) = -4\sin t\mathbf{j} - 4\cos t\mathbf{k}$.
 $a_T = -16\cos t\sin t + 16\cos t\sin t = 0$ and
 $a_N = \frac{1}{\sqrt{17}} |-16\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}|$
 $= \frac{1}{\sqrt{17}} \sqrt{256 + 16} = 4$

12. $\mathbf{r}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{9t^4 + 4t^2 + 1}$,
 $\mathbf{r}''(t) = 6t\mathbf{i} + 2\mathbf{j}$. Thus $a_T = \frac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2 + 1}}$ and
 $a_N = \frac{|-2\mathbf{i} + 6t\mathbf{j} + (6t^2 - 12t^2)\mathbf{k}|}{\sqrt{9t^4 + 4t^2 + 1}}$
 $= \frac{\sqrt{4 + 36t^2 + 36t^4}}{\sqrt{9t^4 + 4t^2 + 1}} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{9t^4 + 4t^2 + 1}}$