11.1 FUNCTIONS OF SEVERAL VARIABLES

A Click here for answers.

S Click here for solutions.

1. If $f(x, y) = x^2 - y^2 + 4xy - 7x + 10$, find
(a) $f(2, 1)$ (b) $f(-3, 5)$
(c) $f(x + h, y)$ (d) $f(x, y + k)$
(e) $f(x, x)$

2. If $g(x, y) = \ln(xy + y - 1)$, find
(a) $g(1, 1)$ (b) $g(e, 1)$
(c) $g(x, 1)$ (d) $g(x + h, y)$
(e) $g(x, y + k)$

3. If $F(x, y) = 3xy/(x^2 + 2y^2)$, find
(a) $F(1, 1)$ (b) $F(-1, 2)$
(c) $F(t, 1)$ (d) $F(-1, y)$
(e) $F(x, x^2)$

4. If $G(x, y, z) = x \sin y \cos z$, find
(a) $G(2, \pi/6, \pi/3)$ (b) $G(4, \pi/4, 0)$
(c) $G(t, t, t)$ (d) $G(u, v, 0)$
(e) $G(x, x + y, x)$

5–10 ▪ Find the domain and range of the function.

5. $f(x, y) = x + 2y - 5$

6. $f(x, y) = \sqrt{x - y}$

7. $f(x, y) = 2/(x + y)$

8. $f(x, y) = \tan^{-1}(y/x)$

9. $f(x, y, z) = x/(yz)$

10. $f(x, y, z) = x \sin(y + z)$

11. Let $f(x, y) = e^{x^2 - y}$.
(a) Evaluate $f(2, 4)$.
(b) Find the domain of f.
(c) Find the range of f.

12. Let $g(x, y) = \sqrt{36 - 9x^2 - 4y^2}$.
(a) Evaluate $g(1, 2)$.
(b) Find and sketch the domain of g.
(c) Find the range of g.

13. Let $f(x, y, z) = x^2 \ln(x - y + z)$.
(a) Evaluate $f(3, 6, 4)$.
(b) Find the domain of f.
(c) Find the range of f.

14. Let $f(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2 - 1}$.
(a) Evaluate $f(1, 3, -4)$.
(b) Find the domain of f.
(c) Find the range of f.

15–25 ▪ Find and sketch the domain of the function.

15. $f(x, y) = xy\sqrt{x^2 + y}$

16. $f(x, y) = \dfrac{\sqrt{9 - x^2 - y^2}}{x + 2y}$

17. $f(x, y) = \dfrac{x^2 + y^2}{x^2 - y^2}$

18. $f(x, y) = \tan(x - y)$

19. $f(x, y) = \ln(xy - 1)$

20. $f(x, y) = \ln(x^2 - y^2)$

21. $f(x, y) = x^2 \sec y$

22. $f(x, y) = \sin^{-1}(x + y)$

23. $f(x, y) = \sqrt{4 - 2x^2 - y^2}$

24. $f(x, y) = \ln x + \ln \sin y$

25. $f(x, y) = \sqrt{y - x} \ln(y + x)$

26–33 ▪ Sketch the graph of the function.

26. $f(x, y) = x$

27. $f(x, y) = \sin y$

28. $f(x, y) = x^2 + 9y^2$

29. $f(x, y) = y^2$

30. $f(x, y) = \sqrt{16 - x^2 - 16y^2}$

31. $f(x, y) = y^2 - x^2$

32. $f(x, y) = 1 - x^2$

33. $f(x, y) = x^2 + y^2 - 4x - 2y + 5$

34–39 ▪ Draw a contour map of the function showing several level curves.

34. $f(x, y) = \dfrac{x}{y}$

35. $f(x, y) = \dfrac{x + y}{x - y}$

36. $f(x, y) = y - \cos x$

37. $f(x, y) = e^{1/(x^2 + y^2)}$

38. $f(x, y) = x^2 + 9y^2$

39. $f(x, y) = e^{xy}$

11.1 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. (a) 7
(b) -45
(c) $x^2+2xh+h^2-y^2+4xy+4hy-7x-7h+10$
(d) $x^2-y^2-2ky-k^2+4xy+4xk-7x+10$
(e) $4x^2-7x+10$

2. (a) 0
(b) $\ln e=1$
(c) $\ln x$
(d) $\ln(xy+hy+y-1)$
(e) $\ln(xy+kx+y+k-1)$

3. (a) 1
(b) $-\frac{2}{3}$
(c) $\dfrac{3t}{t^2+2}$
(d) $-\dfrac{3y}{1+2y^2}$
(e) $\dfrac{3x}{1+2x^2}$

4. (a) $\frac{1}{2}$
(b) $2\sqrt{2}$
(c) $t\sin t\cos t$
(d) $u\sin v$
(e) $x\cos x\,[\sin x\cos y+\sin y\cos x]$

5. $\mathbb{R}^2$, $\mathbb{R}$

6. $\{(x,y)\mid x\geq y\}$, $\{z\mid z\geq 0\}$

7. $\{(x,y)\mid x+y\neq 0\}$, $\{z\mid z\neq 0\}$

8. $\{(x,y)\mid x\neq 0\}$, $\{z\mid -\frac{\pi}{2}<z<\frac{\pi}{2}\}$

9. $\{(x,y,z)\mid yz\neq 0\}$, $\mathbb{R}$

10. $\mathbb{R}^3$, $\mathbb{R}$

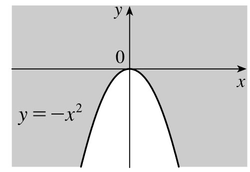
11. (a) 1
(b) $\mathbb{R}^2$
(c) $\{z\mid z>0\}$

12. (a) $\sqrt{11}$
(b) $\left\{(x,y)\mid \frac{1}{4}x^2+\frac{1}{9}y^2\leq 1\right\}$
(c) $\{z\mid 0\leq z\leq 6\}$

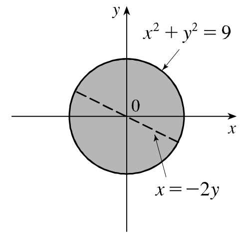
13. (a) 0
(b) $\{(x,y,z)\mid x+z>y\}$
(c) $\mathbb{R}$

14. (a) $\frac{1}{5}$
(b) $\left\{(x,y,z)\mid x^2+y^2+z^2>1\right\}$
(c) $(0,\infty)$

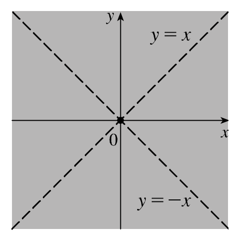
15. $\left\{(x,y)\mid y\geq -x^2\right\}$



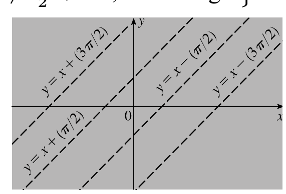
16. $\left\{(x,y)\mid y\neq -\frac{1}{2}x \text{ and } x^2+y^2\leq 9\right\}$



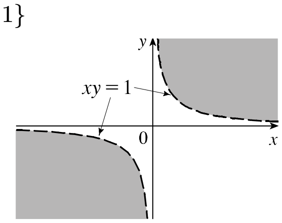
17. $\{(x,y)\mid y\neq x \text{ and } y\neq -x\}$



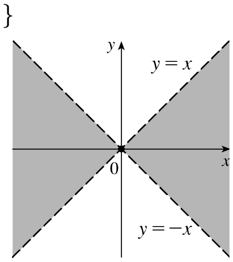
18. $\left\{(x,y)\mid x-y\neq \frac{\pi}{2}+n\pi,\ n \text{ an integer}\right\}$



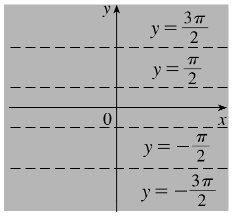
19. $\{(x,y)\mid xy>1\}$



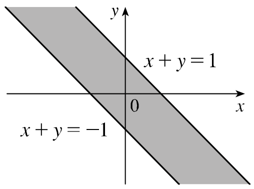
20. $\{(x,y)\mid |y|<|x|\}$



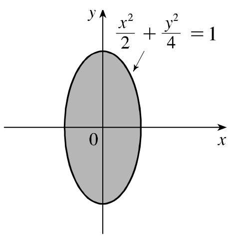
21. $\{(x, y) \mid y \neq \frac{\pi}{2} + n\pi, n \text{ an integer}\}$



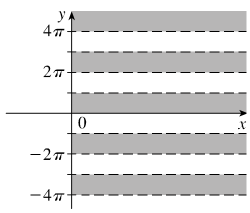
22. $\{(x, y) \mid -1 - x \leq y \text{ and } y \leq 1 - x\}$



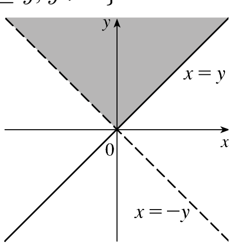
23. $\{(x, y) \mid 2x^2 + y^2 \leq 4\}$



24. $\{(x, y) \mid x > 0 \text{ and } 2n\pi < y < (2n + 1)\pi, n \text{ an integer}\}$



25. $\{(x, y) \mid -y < x \leq y, y > 0\}$



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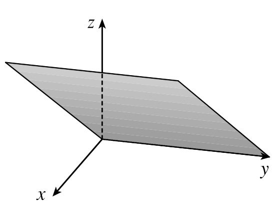

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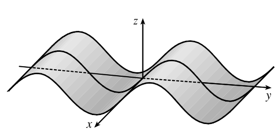

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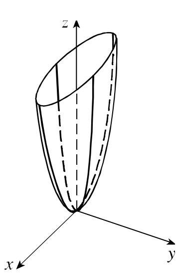

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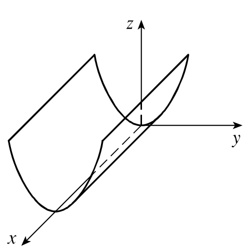

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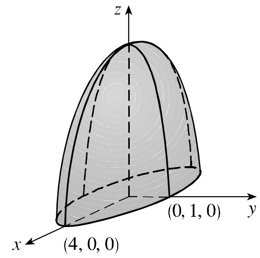

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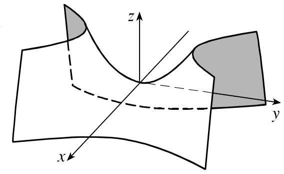

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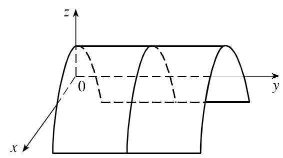

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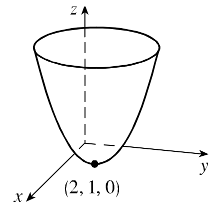

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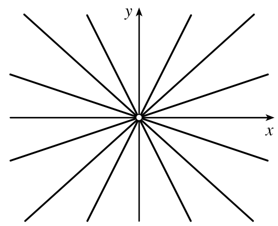

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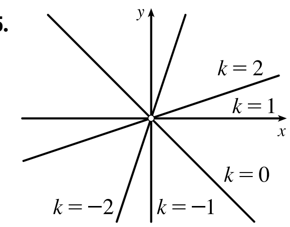

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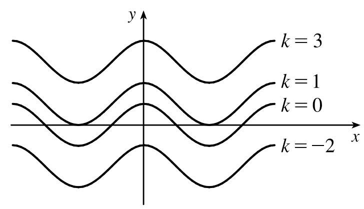

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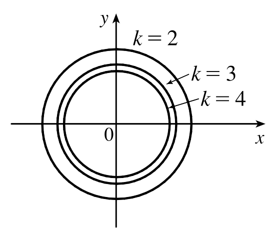

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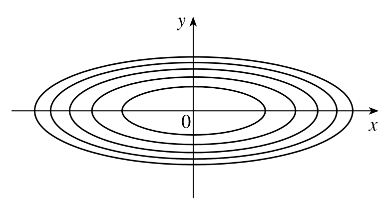

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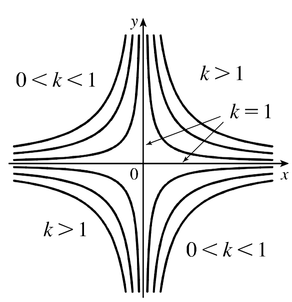


11.1 SOLUTIONS

E Click here for exercises.

1. (a) $f(2,1) = 4 - 1 + 4(2)(1) - 7(2) + 10 = 7$

(b) $f(-3,5) = 9 - 25 + 4(-3)(5) + 21 + 10 = -45$

(c) $f(x+h,y) = (x+h)^2 - y^2 + 4(x+h)y - 7(x+h) + 10$
$= x^2 + 2xh + h^2 - y^2 + 4xy + 4hy - 7x - 7h + 10$

(d) $f(x,y+k) = x^2 - (y+k)^2 + 4x(y+k) - 7x + 10$
$= x^2 - y^2 - 2ky - k^2 + 4xy + 4xk - 7x + 10$

(e) $f(x,x) = x^2 - x^2 + 4x^2 - 7x + 10 = 4x^2 - 7x + 10$

2. (a) $g(1,1) = \ln(1+1-1) = 0$

(b) $g(e,1) = \ln(e+1-1) = \ln e = 1$

(c) $g(x,1) = \ln(x+1-1) = \ln x$

(d) $g(x+h,y) = \ln((x+h)y + y - 1) = \ln(xy + hy + y - 1)$

(e) $g(x,y+k) = \ln(x(y+k) + (y+k) - 1) = \ln(xy + kx + y + k - 1)$

3. (a) $F(1,1) = 3(1)(1)/(1+2) = 1$

(b) $F(-1,2) = 3(-1)(2)/(1+8) = -\frac{2}{3}$

(c) $F(t,1) = \dfrac{3t}{t^2+2}$

(d) $F(-1,y) = \dfrac{3(-1)y}{1+2y^2} = -\dfrac{3y}{1+2y^2}$

(e) $F(x,x^2) = \dfrac{3xx^2}{x^2+2(x^2)^2} = \dfrac{3x^3}{x^2+2x^4} = \dfrac{3x}{1+2x^2}$

4. (a) $G\left(2,\frac{\pi}{6},\frac{\pi}{3}\right) = 2\sin\frac{\pi}{6}\cos\frac{\pi}{3} = \frac{1}{2}$

(b) $G\left(4,\frac{\pi}{4},0\right) = 4\left(\frac{1}{\sqrt{2}}\right)(1) = 2\sqrt{2}$

(c) $G(t,t,t) = t\sin t\cos t$

(d) $G(u,v,0) = u\sin v\cos 0 = u\sin v$

(e) $G(x,x+y,x) = x\sin(x+y)\cos x = x\cos x\left[\sin x\cos y + \sin y\cos x\right]$

5. $D = \mathbb{R}^2$ and the range is $\mathbb{R}$.

6. $D = \{(x,y) \mid x - y \geq 0\} = \{(x,y) \mid x \geq y\}$. Range is $\{z \mid z \geq 0\}$.

7. $x + y \neq 0$ so $D = \{(x,y) \mid x + y \neq 0\}$. Since $2/(x+y)$ can't be zero, the range is $\{z \mid z \neq 0\}$.

8. y/x is defined whenever $x \neq 0$, so $D = \{(x,y) \mid x \neq 0\}$, while the range of the inverse tangent function is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ or $\left\{z \mid -\frac{\pi}{2} < z < \frac{\pi}{2}\right\}$.

9. $D = \{(x,y,z) \mid yz \neq 0\}$ and the range is $\mathbb{R}$.

10. $D = \mathbb{R}^3$ and the range is $\mathbb{R}$.

11. (a) $f(2,4) = e^{2^2-4} = e^0 = 1$.

(b) The exponential function is defined everywhere, so no matter what values of x and y we use, e^{x^2-y} is defined. So the domain of f is $\mathbb{R}^2$.

(c) Because the range of $g(x,y) = x^2 - y$ is $\mathbb{R}$, and the range of e^x is $(0,\infty)$, the range of $e^{g(x,y)} = e^{x^2-y}$ is $\{z \mid z > 0\}$.

12. (a) $g(1,2) = \sqrt{36 - 9(1)^2 - 4(2)^2} = \sqrt{11}$

(b) For the square root to be defined, we need $36 - 9x^2 - 4y^2 \geq 0$ or $\frac{1}{4}x^2 + \frac{1}{9}y^2 \leq 1$. Thus the domain is $\left\{(x,y) \mid \frac{1}{4}x^2 + \frac{1}{9}y^2 \leq 1\right\}$, the points on or inside the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

(c) Since $0 \leq \sqrt{36 - 9x^2 - 4y^2} \leq 6$, the range is $\{z \mid 0 \leq z \leq 6\}$.

13. (a) $f(3,6,4) = 3^2\ln(3-6+4) = 9\ln 1 = 0$.

(b) For the logarithmic function to be defined, we need $x - y + z > 0$. Thus the domain of f is $\{(x,y,z) \mid x + z > y\}$.

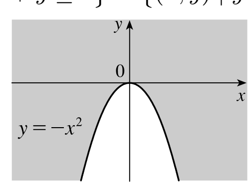
(c) Since $x^2\ln(x-y+z)$ can be any real number, the range of f is $\mathbb{R}$.

14. (a) $f(1,3,-4) = \dfrac{1}{\sqrt{1^2 + 3^2 + (-4)^2 - 1}} = \dfrac{1}{\sqrt{25}} = \dfrac{1}{5}$

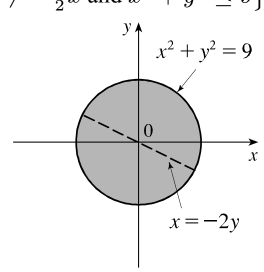
(b) The domain of f is $\left\{(x,y,z) \mid x^2 + y^2 + z^2 > 1\right\}$, the exterior of the sphere $x^2 + y^2 + z^2 = 1$.

(c) Since $\sqrt{x^2 + y^2 + z^2 - 1} > 0$, the range of f is $(0,\infty)$.

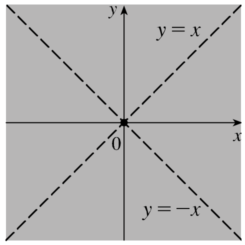
15. $D = \left\{(x,y) \mid x^2 + y \geq 0\right\} = \left\{(x,y) \mid y \geq -x^2\right\}$



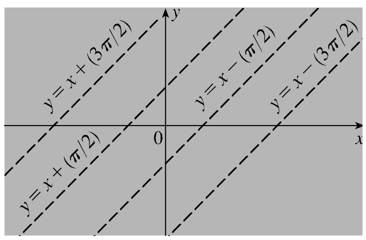
16. $x + 2y \neq 0$ and $9 - x^2 - y^2 \geq 0$, so $D = \left\{(x,y) \mid y \neq -\frac{1}{2}x \text{ and } x^2 + y^2 \leq 9\right\}$.



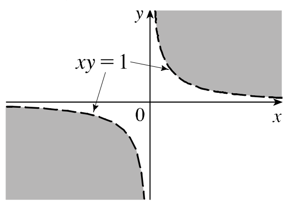
17. $D = \{(x, y) \mid x^2 \neq y^2\} = \{(x, y) \mid y \neq x \text{ and } y \neq -x\}$



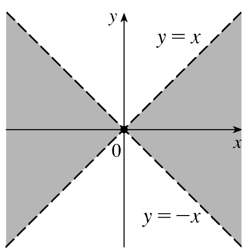
18. $D = \{(x, y) \mid x - y \neq \frac{\pi}{2} + n\pi, n \text{ an integer}\}$



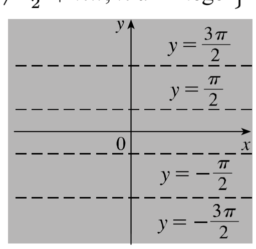
19. $D = \{(x, y) \mid xy > 1\}$



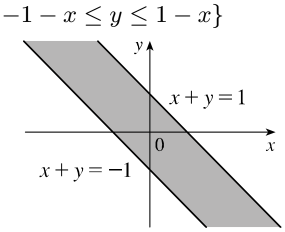
20. $D = \{(x, y) \mid x^2 - y^2 > 0\} = \{(x, y) \mid |y| < |x|\}$



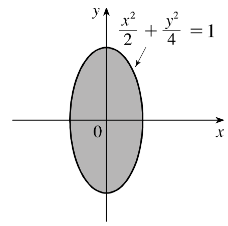
21. $D = \{(x, y) \mid y \neq \frac{\pi}{2} + n\pi, n \text{ an integer}\}$



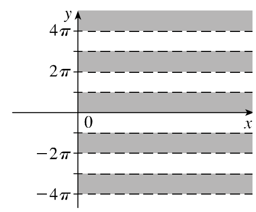
22. $D = \{(x, y) \mid -1 \leq x + y \leq 1\}$
$= \{(x, y) \mid -1 - x \leq y \leq 1 - x\}$



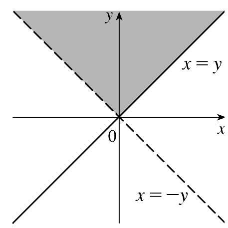
23. $D = \{(x, y) \mid 2x^2 + y^2 \leq 4\}$, the points on or inside the ellipse $\frac{1}{2}x^2 + \frac{1}{4}y^2 = 1$.



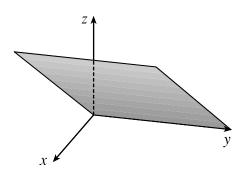
24. Since $\sin y > 0$ implies $2n\pi < y < (2n+1)\pi$, n an integer,
$D = \{(x, y) \mid x > 0 \text{ and } 2n\pi < y < (2n+1)\pi,$
$n \text{ an integer}\}$.



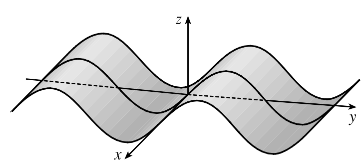
25. We need $y - x \geq 0$ or $y \geq x$ and $y + x > 0$ or $x > -y$. Thus $D = \{(x, y) \mid -y < x \leq y, y > 0\}$.



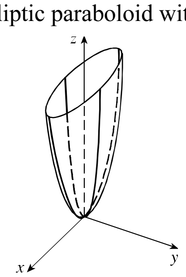
26. $z = x$, a plane which intersects the xz-plane in the line $z = x$, $y = 0$. The portion of this plane that lies in the first octant is shown.



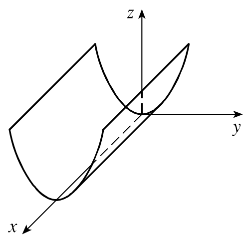
27. $z = \sin y$, a "wave."



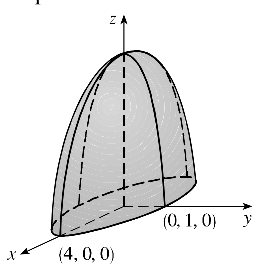
28. $z = x^2 + 9y^2$, an elliptic paraboloid with vertex the origin.



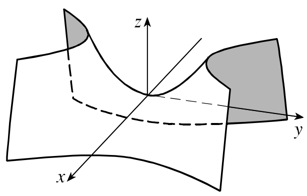
29. $z = y^2$, a parabolic cylinder.



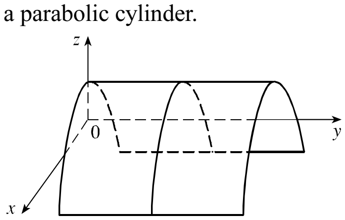
30. $z = \sqrt{16 - x^2 - 16y^2}$ so $z \geq 0$ and $z^2 + x^2 + 16y^2 = 16$, the top half of an ellipsoid.



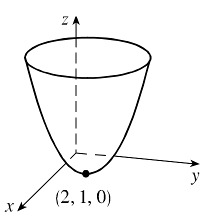
31. $z = y^2 - x^2$, a hyperbolic paraboloid.



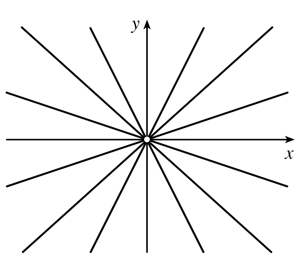
32. $z = 1 - x^2$, a parabolic cylinder.



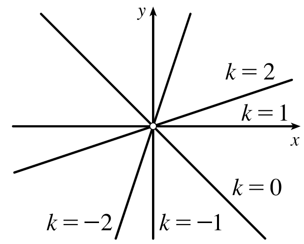
33. $z = x^2 + y^2 - 4x - 2y + 5$ or, completing the squares, $z = (x-2)^2 + (y-1)^2$, a circular paraboloid with vertex at $(2, 1, 0)$.



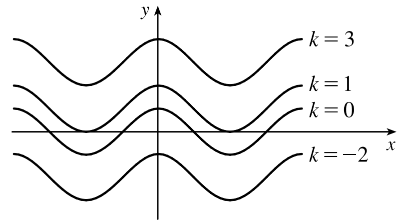
34. $k = x/y$ or $x = ky$ is a family of lines without the point $(0, 0)$.



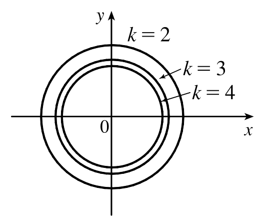
35. $k = \dfrac{x+y}{x-y}$ is a family of lines with slope $\dfrac{k-1}{k+1}$ (for $k \neq -1$) without the origin. For $k = -1$, the curve is the y-axis without the origin.



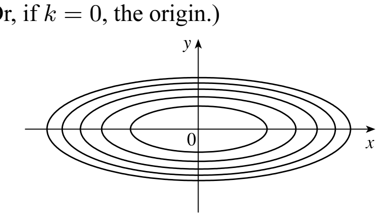
36. $k = y - \cos x$ or $y = k + \cos x$



37. $k = e^{1/(x^2+y^2)}$, thus $k > 1$ and $1/(x^2 + y^2) = \ln k$ or $x^2 + y^2 = 1/\ln k$, a family of circles.



38. $k = x^2 + 9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k = 0$, the origin.)



39. $k = e^{xy}$ where here k must be positive or we get the empty set. Since $k = e^{xy}$, $xy = \ln k$. Thus for $k = 1$, the curves are the coordinate axes; for $0 < k < 1$, hyperbolas in the second and fourth quadrants; for $k > 1$ the curves are hyperbolas in the first and third quadrants.