11.4 TANGENT PLANES AND LINEAR APPROXIMATIONS

A Click here for answers.

S Click here for solutions.

1–9 ▪ Find an equation of the tangent plane to the given surface at the specified point.

1. $z = x^2 + 4y^2$, $(2, 1, 8)$

2. $z = x^2 - y^2$, $(3, -2, 5)$

3. $z = 5 + (x - 1)^2 + (y + 2)^2$, $(2, 0, 10)$

4. $z = xy$, $(-1, 2, -2)$

5. $z = \sqrt{x - y}$, $(5, 1, 2)$

6. $z = y^2 - x^2$, $(-4, 5, 9)$

7. $z = \sin(x + y)$, $(1, -1, 0)$

8. $z = \ln(2x + y)$, $(-1, 3, 0)$

9. $z = e^x \ln y$, $(3, 1, 0)$

10–11 ▪ Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

10. $z = xy$, $(-1, 2, -2)$

11. $z = \sqrt{x - y}$, $(5, 1, 2)$

12–13 ▪ Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

12. $f(x, y) = y \ln x$, $(2, 1)$

13. $f(x, y) = \sqrt{1 + x^2 y^2}$, $(0, 2)$

14–22 ▪ Find the differential of the function.

14. $z = x^2 y^3$

15. $v = \ln(2x - 3y)$

16. $w = x \sin yz$

17. $z = x^4 - 5x^2 y + 6xy^3 + 10$

18. $z = \dfrac{1}{x^2 + y^2}$

19. $z = ye^{xy}$

20. $u = e^x \cos xy$

21. $w = x^2 y + y^2 z$

22. $w = \dfrac{x + y}{y + z}$

23–26 ▪ Use differentials to approximate the value of f at the given point.

23. $f(x, y) = \sqrt{20 - x^2 - 7y^2}$, $(1.95, 1.08)$

24. $f(x, y) = \ln(x - 3y)$, $(6.9, 2.06)$

25. $f(x, y, z) = x^2 y^3 z^4$, $(1.05, 0.9, 3.01)$

26. $f(x, y, z) = xy^2 \sin \pi z$, $(3.99, 4.98, 4.03)$

27–30 ▪ Use differentials to approximate the number.

27. $8.94\sqrt{9.99 - (1.01)^3}$

28. $\left(\sqrt{99} + \sqrt[3]{124}\right)^4$

29. $\sqrt{0.99}\, e^{0.02}$

30. $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$

11.4 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $4x + 8y - z = 8$

2. $6x + 4y - z = 5$

3. $2x + 4y - z = -6$

4. $2x - y - z = -2$

5. $x - y - 4z = -4$

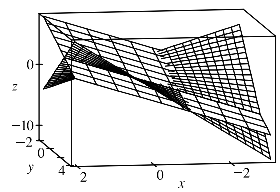
6. $z = 8x + 10y - 9$

7. $z = x + y$

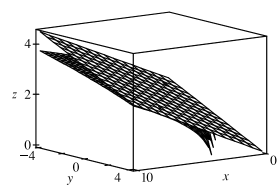
8. $z = 2x + y - 1$

9. $z = e^3 y - e^3$

10.



11.



12. $\frac{1}{2}x + (\ln 2)\, y - 1$

13. 1

14. $2xy^3\, dx + 3x^2y^2\, dy$

15. $\dfrac{1}{2x - 3y}\,(2\, dx - 3\, dy)$

16. $(\sin yz)\, dx + (xz \cos yz)\, dy + (xy \cos yz)\, dz$

17. $\left(4x^3 - 10xy + 6y^3\right) dx + \left(-5x^2 + 18xy^2\right) dy$

18. $-\dfrac{2}{(x^2 + y^2)^2}\,(x\, dx + y\, dy)$

19. $y^2 e^{xy}\, dx + e^{xy}\,(1 + xy)\, dy$

20. $e^x\,(\cos xy - y \sin xy)\, dx - (xe^x \sin xy)\, dy$

21. $2xy\, dx + \left(x^2 + 2yz\right) dy + y^2\, dz$

22. $\dfrac{(y + z)\, dx + (z - x)\, dy - (x + y)\, dz}{(y + z)^2}$

23. $2.84\overline{6}$

24. -0.28

25. 65.88

26. 3π

27. 26.76

28. 49,770

29. 1.015

30. 6.9914

11.4 SOLUTIONS

E Click here for exercises.

1. $z = f(x,y) = x^2 + 4y^2 \quad \Rightarrow \quad f_x(x,y) = 2x$, $f_y(x,y) = 8y$, $f_x(2,1) = 4$, $f_y(2,1) = 8$. Thus the equation of the tangent plane is $z - 8 = 4(x-2) + 8(y-1)$ or $4x + 8y - z = 8$.

2. $z = f(x,y) = x^2 - y^2 \quad \Rightarrow \quad f_x(x,y) = 2x$, $f_y(x,y) = -2y$, $f_x(3,-2) = 6$, $f_y(3,-2) = 4$. Thus the equation is $z - 5 = 6(x-3) + 4(y+2)$ or $6x + 4y - z = 5$.

3. $z = f(x,y) = 5 + (x-1)^2 + (y+2)^2 \quad \Rightarrow$ $f_x(x,y) = 2(x-1)$, $f_y(x,y) = 2(y+2)$, $f_x(2,0) = 2$, $f_y(2,0) = 4$ and the equation is $z - 10 = 2(x-2) + 4y$ or $2x + 4y - z = -6$.

4. $f_x(-1,2) = 2$ and $f_y(-1,2) = -1$, so an equation of the tangent plane is $z + 2 = 2(x+1) + (-1)(y-2)$ or $2x - y - z = -2$.

5. $f_x(x,y) = \frac{1}{2}(x-y)^{-1/2}$, $f_x(5,1) = \frac{1}{4}$, $f_y(x,y) = -\frac{1}{2}(x-y)^{-1/2}$, and $f_y(5,1) = -\frac{1}{4}$, so an equation of the tangent plane is $z - 2 = \frac{1}{4}(x-5) - \frac{1}{4}(y-1)$ or $x - y - 4z = -4$.

6. $z = f(x,y) = y^2 - x^2 \quad \Rightarrow \quad f_x(x,y) = -2x$, $f_y(x,y) = 2y$, so $f_x(-4,5) = 8$, $f_y(-4,5) = 10$. By Equation 2, an equation of the tangent plane is

$z - 9 = f_x(-4,5)[x - (-4)] + f_y(-4,5)(y-5) \quad \Rightarrow$

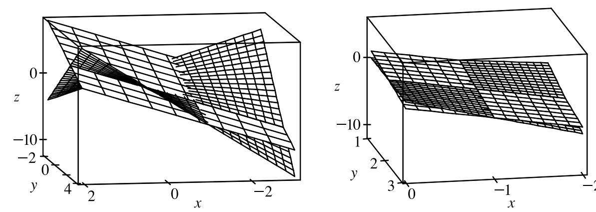
$z - 9 = 8(x+4) + 10(y-5)$ or $z = 8x + 10y - 9$.

7. $z = f(x,y) = \sin(x+y) \quad \Rightarrow \quad f_x(x,y) = \cos(x+y)$, $f_y(x,y) = \cos(x+y)$, $f_x(1,-1) = 1 = f_y(1,-1)$ and an equation of the tangent plane is $z = (x-1) + (y+1)$ or $z = x + y$.

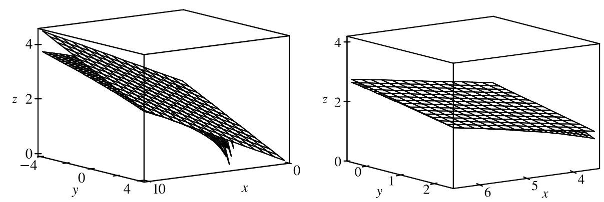
8. $z = f(x,y) = \ln(2x+y) \quad \Rightarrow \quad f_x(x,y) = \dfrac{2}{2x+y}$, $f_y(x,y) = \dfrac{1}{2x+y}$, $f_x(-1,3) = 2$, $f_y(-1,3) = 1$. Thus an equation of the tangent plane is $z = 2(x+1) + (y-3)$ or $z = 2x + y - 1$.

9. $z = f(x,y) = e^x \ln y \quad \Rightarrow \quad f_x(x,y) = e^x \ln y$, $f_y(x,y) = e^x/y$, $f_x(3,1) = 0$, $f_y(3,1) = e^3$, and an equation of the tangent plane is $z = e^3(y-1)$ or $z = e^3 y - e^3$.

10. $z = f(x,y) = xy$, so $f_x(x,y) = y \quad \Rightarrow \quad f_x(-1,2) = 2$, $f_y(x,y) = x \quad \Rightarrow \quad f_y(-1,2) = -1$ and an equation of the tangent plane is $z + 2 = 2(x+1) + (-1)(y-2)$ or $z = 2x - y + 2$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane appear to coincide.



11. $z = f(x,y) = \sqrt{x-y}$, so $f_x(x,y) = \frac{1}{2}(x-y)^{-1/2}$, $f_x(5,1) = \frac{1}{4}$, $f_y(x,y) = -\frac{1}{2}(x-y)^{-1/2}$, $f_y(5,1) = -\frac{1}{4}$, and an equation of the tangent plane is $z - 2 = \frac{1}{4}(x-5) - \frac{1}{4}(y-1)$ or $z = \frac{1}{4}x - \frac{1}{4}y + 1$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane appear to coincide.



12. $f(x,y) = y \ln x$. The partial derivatives are $f_x(x,y) = y/x$ and $f_y(x,y) = \ln x$, so $f_x(2,1) = \frac{1}{2}$ and $f_y(2,1) = \ln 2$. Both f_x and f_y are continuous functions for $x > 0$, so f is differentiable at $(2,1)$ by Theorem 8. The linearization of f at $(2,1)$ is given by

$$\begin{aligned} L(x,y) &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= \ln 2 + \tfrac{1}{2}(x-2) + \ln 2\,(y-1) \\ &= \tfrac{1}{2}x + (\ln 2)\,y - 1 \end{aligned}$$

13. $f(x,y) = \sqrt{1 + x^2y^2}$. The partial derivatives are

$f_x(x,y) = \frac{1}{2}\left(1 + x^2y^2\right)^{-1/2}\left(2xy^2\right) = \dfrac{xy^2}{\sqrt{1+x^2y^2}}$ and

$f_y(x,y) = \frac{1}{2}\left(1 + x^2y^2\right)^{-1/2}\left(2x^2y\right) = \dfrac{x^2y}{\sqrt{1+x^2y^2}}$, so

$f_x(0,2) = 0$ and $f_y(0,2) = 0$. Both f_x and f_y are continuous functions, so f is differentiable at $(0,2)$, and the linearization of f at $(0,2)$ is

$$\begin{aligned} L(x,y) &= f(0,2) + f_x(0,2)(x-0) + f_y(0,2)(y-2) \\ &= 1 + 0(x) + 0(y-2) = 1 \end{aligned}$$

14. $z = x^2y^3 \quad \Rightarrow$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = 2xy^3\,dx + 3x^2y^2\,dy$$

15. $v = \ln(2x - 3y) \quad \Rightarrow$

$$dv = \left(\frac{2}{2x-3y}\right) dx + \left(\frac{-3}{2x-3y}\right) dy = \frac{1}{2x-3y}\,(2\,dx - 3\,dy)$$

16. $w = x \sin yz \quad \Rightarrow$

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz = (\sin yz)\,dx + (xz\cos yz)\,dy + (xy\cos yz)\,dz$$

17. $dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy$

$$= \left(4x^3 - 10xy + 6y^3\right) dx + \left(-5x^2 + 18xy^2\right) dy$$

18. $dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy$

$$= -\frac{2}{(x^2+y^2)^2}\,(x\,dx + y\,dy)$$

19. $z = ye^{xy} \quad \Rightarrow$

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = y^2 e^{xy}\,dx + e^{xy}\,(1 + xy)\,dy$$

20. $u = e^x \cos xy \quad \Rightarrow$

$$du = e^x\,(\cos xy - y\sin xy)\,dx - (xe^x \sin xy)\,dy$$

21. $w = x^2 y + y^2 z \quad \Rightarrow$

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz = 2xy\,dx + \left(x^2 + 2yz\right) dy + y^2\,dz$$

22. $w = \dfrac{x+y}{y+z} \quad \Rightarrow$

$$dw = \frac{dx}{y+z} + \frac{[(y+z) - (x+y)]\,dy}{(y+z)^2} - \frac{(x+y)\,dz}{(y+z)^2} = \frac{(y+z)\,dx + (z-x)\,dy - (x+y)\,dz}{(y+z)^2}$$

23. $f(x,y) = \sqrt{20 - x^2 - 7y^2} \quad \Rightarrow$

$f_x = -\dfrac{x}{\sqrt{20 - x^2 - 7y^2}}$ and $f_y = -\dfrac{7y}{\sqrt{20 - x^2 - 7y^2}}$.

Since $f(2,1) = \sqrt{20 - 4 - 7} = 3$, we set $(a,b) = (2,1)$. Then $\Delta x = -0.05$, $\Delta y = 0.08$. Thus

$$f(1.95, 1.08) \approx f(2,1) + dz = 3 + \left(-\tfrac{2}{3}\right)(-0.05) + \left(-\tfrac{7}{3}\right)(0.08) = 2.84\overline{6}$$

24. $f(x,y) = \ln(x - 3y) \quad \Rightarrow \quad f_x = \dfrac{1}{x-3y}$ and $f_y = -\dfrac{3}{x-3y}$. Since $f(7,2) = \ln(7-6) = 0$, we set $(a,b) = (7,2)$. Then $\Delta x = -0.1$ and $\Delta y = 0.06$, so

$$f(6.9, 2.06) \approx f(7,2) + dz = 0 + (1)(-0.1) + (-3)(0.06) = -0.28$$

25. $f(x,y,z) = x^2y^3z^4 \quad \Rightarrow \quad f_x = 2xy^3z^4$, $f_y = 3x^2y^2z^4$ and $f_z = 4x^2y^3z^3$. Since $f(1,1,3) = 81$, we set $(a,b,c) = (1,1,3)$. Then $\Delta x = 0.05$, $\Delta y = -0.1$, $\Delta z = 0.01$, and so

$$f(1.05, 0.9, 3.01) \approx f(1,1,3) + dw = 81 + (162)(0.05) + (243)(-0.1) + (108)(0.01) = 65.88$$

26. $f(x,y,z) = xy^2 \sin \pi z \quad \Rightarrow \quad f_x = y^2 \sin \pi z$, $f_y = 2xy\sin\pi z$, $f_z = \pi xy^2 \cos \pi z$. Since $f(4,5,4) = 0$, we set $(a,b,c) = (4,5,4)$. Then $\Delta x = -0.01$, $\Delta y = -0.02$, and $\Delta z = 0.03$, so

$$f(3.99, 4.98, 4.03) \approx f(4,5,4) + dw = 0 + (0)(-0.01) + (0)(-0.02) + (100\pi)(0.03) = 3\pi \approx 9.4248$$

27. Let $w = f(x,y,z) = x\sqrt{y - z^3} \quad \Rightarrow \quad f_x = \sqrt{y - z^3}$, $f_y = \dfrac{x}{2\sqrt{y - z^3}}$, and $f_z = -\dfrac{3xz^2}{2\sqrt{y - z^3}}$. Then $f(9,10,1) = 27$, so we set $(a,b,c) = (9,10,1)$. Then $\Delta x = -0.06$, $\Delta y = -0.01$, and $\Delta z = 0.01$. Thus

$$8.94\sqrt{9.99 - (1.01)^3} \approx 27 + (3)(-0.06) + \tfrac{9}{6}(-0.01) + \left(-\tfrac{27}{6}\right)(0.01) = 26.76$$

28. Let $z = f(x,y) = \left(\sqrt{x} + \sqrt[3]{y}\right)^4 \quad \Rightarrow \quad f_x = 2\frac{\left(\sqrt{x} + \sqrt[3]{y}\right)^3}{\sqrt{x}}$, $f_y = 4\frac{\left(\sqrt{x} + \sqrt[3]{y}\right)^3}{3y^{2/3}}$. Then $f(100, 125) = (10 + 5)^4 = 50{,}625$. Set $(a,b) = (100, 125)$, so $\Delta x = -1$, $\Delta y = -1$. Thus

$$\left(\sqrt{99} + \sqrt[3]{124}\right)^4 \approx 50{,}625 + \frac{2 \cdot 3375}{10}(-1) + \frac{4 \cdot 3375}{75}(-1) = 49{,}770$$

29. Let $z = f(x,y) = \sqrt{x}e^y \quad \Rightarrow \quad f_x = \dfrac{e^y}{2\sqrt{x}}$, $f_y = \sqrt{x}e^y$. Now $f(1,0) = 1$, so we set $(a,b) = (1,0)$, $\Delta x = -0.01$, $\Delta y = 0.02$. Thus $\sqrt{0.99}\,e^{0.02} \approx 1 + \frac{1}{2}(-0.01) + 1\,(0.02) = 1.015$.

30. Let $w = f(x,y,z) = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow$ $f_x = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$, $f_z = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$. Now $f(3,2,6) = \sqrt{49} = 7$, so we set $(a,b,c) = (3,2,6)$, $\Delta x = 0.02$, $\Delta y = -0.03$, and $\Delta z = -0.01$. Thus

$$\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} \approx f(3,2,6) + dw = 7 + \tfrac{3}{7}(0.02) + \tfrac{2}{7}(-0.03) + \tfrac{6}{7}(-0.01) \approx 6.9914$$