12.1 DOUBLE INTEGRALS OVER RECTANGLES

A Click here for answers.

S Click here for solutions.

1. Find approximations to $\iint_R (x - 3y^2)\, dA$ using the same subrectangles as in Example 3 but choosing the sample point to be the (a) upper left corner, (b) upper right corner, (c) lower left corner, (d) lower right corner of each subrectangle.

2. Find the approximation to the volume in Example 1 if the Midpoint Rule is used.

3. (a) Estimate the volume of the solid that lies below the surface $z = x^2 + 4y$ and above the rectangle $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 3\}$. Use a Riemann sum with $m = 2$, $n = 3$, and take the sample point to be the upper right corner of each subrectangle.
(b) Use the Midpoint Rule to estimate the volume of the solid in part (a).

4. If $R = [-2, 2] \times [-1, 1]$, use a Riemann sum with $m = n = 4$ to estimate the value of $\iint_R (2x + x^2y)\, dA$. Take the sample points to be the lower left corners of the subrectangles.

5–8 ■ Find $\int_0^2 f(x, y)\, dy$ and $\int_0^1 f(x, y)\, dx$.

5. $f(x, y) = x^2y^3$

6. $f(x, y) = 2xy - 3x^2$

7. $f(x, y) = xe^{x+y}$

8. $f(x, y) = \dfrac{x}{y^2 + 1}$

9–16 ■ Calculate the iterated integral.

9. $\int_0^4 \int_0^2 x\sqrt{y}\, dx\, dy$

10. $\int_0^2 \int_0^3 e^{x-y}\, dy\, dx$

11. $\int_{-1}^1 \int_0^1 (x^3y^3 + 3xy^2)\, dy\, dx$

12. $\int_0^1 \int_1^2 (x^4 - y^2)\, dx\, dy$

13. $\int_0^{\pi/4} \int_0^3 \sin x\, dy\, dx$

14. $\int_0^{\pi/2} \int_0^{\pi/2} \sin x \cos y\, dy\, dx$

15. $\int_0^3 \int_0^1 \sqrt{x + y}\, dx\, dy$

16. $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x + y)\, dy\, dx$

17–23 ■ Calculate the double integral.

17. $\iint_R (2y^2 - 3xy^3)\, dA, \quad R = \{(x, y) \mid 1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 3\}$

18. $\iint_R \left(xy^2 + \dfrac{y}{x}\right) dA, \quad R = \{(x, y) \mid 2 \leqslant x \leqslant 3,\ -1 \leqslant y \leqslant 0\}$

19. $\iint_R x \sin y\, dA, \quad R = \{(x, y) \mid 1 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant \pi/6\}$

20. $\iint_R \dfrac{1 + x}{1 + y}\, dA, \quad R = \{(x, y) \mid -1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}$

21. $\iint_R xye^y\, dA, \quad R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}$

22. $\iint_R xe^{xy}\, dA, \quad R = [0, 1] \times [0, 1]$

23. $\iint_R \dfrac{1}{x + y}\, dA, \quad R = [1, 2] \times [0, 1]$

24. Find the volume of the solid lying under the plane $z = 2x + 5y + 1$ and above the rectangle $\{(x, y) \mid -1 \leqslant x \leqslant 0,\ 1 \leqslant y \leqslant 4\}$.

25. Find the volume of the solid lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3]$.

26. Find the volume of the solid lying under the hyperbolic paraboloid $z = y^2 - x^2$ and above the square $R = [-1, 1] \times [1, 3]$.

27. Find the average value of $f(x, y) = x \sin xy$ over the rectangle $R = [0, \pi/2] \times [0, 1]$.

12.1 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. (a) -17.75
(b) -15.75
(c) -8.75
(d) -6.75

2. 49

3. (a) 63
(b) 43.5

4. -11

5. $4x^2$, $\frac{1}{3}y^3$

6. $4x - 6x^2$, $y - 1$

7. $xe^x\left(e^2 - 1\right)$, e^y

8. $x\tan^{-1}2$, $\dfrac{1}{2\left(y^2+1\right)}$

9. $\frac{32}{3}$

10. $e^2 - e^{-1} - 1 + e^{-3}$

11. 0

12. $\frac{88}{15}$

13. $3\left(1 - \frac{1}{\sqrt{2}}\right)$

14. 1

15. $\frac{4}{15}\left(31 - 9\sqrt{3}\right)$

16. 2

17. $-\frac{585}{8}$

18. $\frac{5}{6} + \ln\sqrt{\frac{2}{3}}$

19. $\frac{15\left(2-\sqrt{3}\right)}{4}$

20. $\frac{9}{2}\ln 2$

21. 2

22. $e - 2$

23. $\ln\frac{27}{16}$

24. $\frac{75}{2}$

25. 104

26. 16

27. $1 - \frac{2}{\pi}$

12.1 SOLUTIONS

E Click here for exercises.

1. (a) $\sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$

$$= f\left(0, \tfrac{3}{2}\right)\Delta A + f(0,2)\,\Delta A + f\left(1, \tfrac{3}{2}\right)\Delta A + f(1,2)\,\Delta A$$
$$= \left(-\tfrac{27}{4}\right)\tfrac{1}{2} + (-12)\,\tfrac{1}{2} + \left(1 - \tfrac{27}{4}\right)\tfrac{1}{2} + (1-12)\,\tfrac{1}{2}$$
$$= -17.75$$

(b) $\frac{1}{2}\left[f\left(1,\tfrac{3}{2}\right) + f(1,2) + f\left(2,\tfrac{3}{2}\right) + f(2,2)\right]$
$$= \tfrac{1}{2}\left[-\tfrac{23}{4} + (-11) + \left(-\tfrac{19}{4}\right) + (-10)\right]$$
$$= \tfrac{1}{2}\left(-\tfrac{63}{2}\right) = -15.75$$

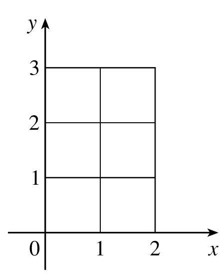
(c) $\frac{1}{2}\left[f(0,1) + f\left(0,\tfrac{3}{2}\right) + f(1,1) + f\left(1,\tfrac{3}{2}\right)\right]$
$$= \tfrac{1}{2}\left[-3 - \tfrac{27}{4} - 2 - \tfrac{23}{4}\right] = -8.75$$

(d) $\frac{1}{2}\left[f(1,1) + f\left(1,\tfrac{3}{2}\right) + f(2,1) + f\left(2,\tfrac{3}{2}\right)\right]$
$$= \tfrac{1}{2}\left[-2 - \tfrac{23}{4} - 1 - \tfrac{19}{4}\right] = -6.75$$

2. $V \approx (1)\left[f\left(\tfrac{1}{2},\tfrac{1}{2}\right) + f\left(\tfrac{1}{2},\tfrac{3}{2}\right) + f\left(\tfrac{3}{2},\tfrac{1}{2}\right) + f\left(\tfrac{3}{2},\tfrac{3}{2}\right)\right]$
$$= \left[\tfrac{61}{4} + \tfrac{45}{4} + \tfrac{53}{4} + \tfrac{37}{4}\right] = 49$$

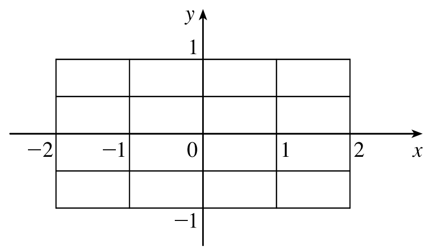
3. (a) The subrectangles are shown in the figure. The surface is the graph of $f(x,y) = x^2 + 4y$ and $\Delta A = 1$, so we estimate

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{3} f(x_i, y_j)\,\Delta A$$
$$= f(1,1)\,\Delta A + f(1,2)\,\Delta A + f(1,3)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A + f(2,3)\,\Delta A$$
$$= 5(1) + 9(1) + 13(1) + 8(1) + 12(1) + 16(1)$$
$$= 63$$



(b) $V \approx \sum_{i=1}^{2}\sum_{j=1}^{3} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$
$$= f\left(\tfrac{1}{2},\tfrac{1}{2}\right)\Delta A + f\left(\tfrac{1}{2},\tfrac{3}{2}\right)\Delta A + f\left(\tfrac{1}{2},\tfrac{5}{2}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{1}{2}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{3}{2}\right)\Delta A + f\left(\tfrac{3}{2},\tfrac{5}{2}\right)\Delta A$$
$$= \tfrac{9}{4}(1) + \tfrac{25}{4}(1) + \tfrac{41}{4}(1) + \tfrac{17}{4}(1) + \tfrac{33}{4}(1) + \tfrac{49}{4}(1)$$
$$= \tfrac{87}{2} = 43.5$$

4. The subrectangles are shown in the figure.



Since $\Delta A = \frac{1}{2}$, we estimate

$$\iint_R \left(2x + x^2 y\right) dA \approx \sum_{i=1}^{4}\sum_{j=1}^{4} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$
$$= \tfrac{1}{2}\Big[f(-2,-1) + f\left(-2,-\tfrac{1}{2}\right) + f(-2,0) + f\left(-2,\tfrac{1}{2}\right) + f(-1,-1) + f\left(-1,-\tfrac{1}{2}\right) + f(-1,0) + f\left(-1,\tfrac{1}{2}\right) + f(0,-1) + f\left(0,-\tfrac{1}{2}\right) + f(0,0) + f\left(0,\tfrac{1}{2}\right) + f(1,-1) + f\left(1,-\tfrac{1}{2}\right) + f(1,0) + f\left(1,\tfrac{1}{2}\right)\Big]$$
$$= \tfrac{1}{2}(-22) = -11$$

5. $\int_0^2 x^2 y^3\,dy = x^2\left[\tfrac{1}{4}y^4\right]_0^2 = 4x^2,$

$\int_0^1 x^2 y^3\,dx = y^3\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}y^3$

6. $\int_0^2 \left(2xy - 3x^2\right) dy = \left[xy^2 - 3x^2 y\right]_0^2 = 4x - 6x^2,$

$\int_0^1 \left(2xy - 3x^2\right) dx = \left[yx^2 - x^3\right]_0^1 = y - 1$

7. $\int_0^2 xe^{x+y}\,dy = xe^x\left[e^y\right]_0^2 = x\left(e^{x+2} - e^x\right)$
$$= xe^x\left(e^2 - 1\right)$$

$\int_0^1 xe^{x+y}\,dx = e^y\int_0^1 xe^x\,dx = e^y\left[xe^x - e^x\right]_0^1 = e^y$

8. $\displaystyle\int_0^2 \frac{x}{y^2+1}\,dy = x\left[\tan^{-1}y\right]_0^2 = x\tan^{-1}2,$

$\displaystyle\int_0^1 \frac{x}{y^2+1}\,dx = \frac{1}{y^2+1}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2\left(y^2+1\right)}$

9. $\int_0^4\int_0^2 x\sqrt{y}\,dx\,dy = \int_0^4 \sqrt{y}\left[\tfrac{1}{2}x^2\right]_0^2 dy = \int_0^4 2\sqrt{y}\,dy$
$$= \left[\tfrac{4}{3}y^{3/2}\right]_0^4 = \tfrac{32}{3}$$

10. $\int_0^2\int_0^3 e^{x-y}\,dy\,dx = \int_0^2 \left[-e^{x-y}\right]_0^3 dx$
$$= \int_0^2 e^x\left(1 - e^{-3}\right) dx = e^2 - e^{-1} - 1 + e^{-3}$$

11. $\int_{-1}^1\int_0^1 \left(x^3y^2 + 3xy^2\right) dy\,dx$
$$= \int_{-1}^1 \left[\tfrac{1}{4}x^3y^4 + xy^3\right]_{y=0}^{y=1} dx = \int_{-1}^1 \left[\tfrac{1}{4}x^3 + x\right] dx$$
$$= \left[\tfrac{1}{16}x^4 + \tfrac{1}{2}x^2\right]_{-1}^1 = 0$$

Alternate Solution: Applying Fubini's Theorem, the integral equals

$\int_0^1\int_{-1}^1 \left(x^3y^2 + 3xy^2\right) dx\,dy$
$$= \int_0^1 \left[\tfrac{1}{4}y^2x^4 + \tfrac{3}{2}y^2x^2\right]_{x=-1}^{x=1} dy = \int_0^1 0\,dy = 0$$

12. $\int_0^1\int_1^2 \left(x^4 - y^2\right) dx\, dy = \int_1^2\int_0^1 \left(x^4 - y^2\right) dy\, dx$
$= \int_1^2 \left[x^4 y - \frac{1}{3}y^3\right]_{y=0}^{y=1} dx = \int_1^2 \left[x^4 - \frac{1}{3}\right] dx$
$= \left[\frac{1}{5}x^5 - \frac{1}{3}x\right]_1^2 = \frac{88}{15}$

13. $\int_0^{\pi/4}\int_0^3 \sin x\, dy\, dx = 3\int_0^{\pi/4} \sin x\, dx = 3\left[-\cos x\right]_0^{\pi/4}$
$= 3\left(1 - \frac{1}{\sqrt{2}}\right)$

14. $\int_0^{\pi/2}\int_0^{\pi/2} \sin x \cos y\, dy\, dx$
$= \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy$ (as in Example 8)
$= \left[-\cos x\right]_0^{\pi/2}\left[\sin y\right]_0^{\pi/2} = -(0-1)(1-0) = 1$

15. $\int_0^3\int_0^1 \sqrt{x+y}\, dx\, dy = \int_0^3 \left[\frac{2}{3}(x+y)^{3/2}\right]_{x=0}^{x=1} dy$
$= \frac{2}{3}\int_0^3 \left[(1+y)^{3/2} - y^{3/2}\right] dy$
$= \frac{2}{3}\left[\frac{2}{5}(1+y)^{5/2} - \frac{2}{5}y^{5/2}\right]_0^3$
$= \frac{4}{15}\left[32 - 3^{5/2} - 1\right] = \frac{4}{15}\left(31 - 9\sqrt{3}\right)$

16. $\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\, dy\, dx$
$= \int_0^{\pi/2} \left[-\cos(x+y)\right]_{y=0}^{y=\pi/2} dx$
$= \int_0^{\pi/2} \left[\cos x - \cos\left(x + \frac{\pi}{2}\right)\right] dx$
$= \left[\sin x - \sin\left(x + \frac{\pi}{2}\right)\right]_0^{\pi/2}$
$= (1-0) - (0-1) = 2$

17. $\int_1^2\int_0^3 \left(2y^2 - 3xy^3\right) dy\, dx = \int_1^2 \left[\frac{2}{3}y^3 - \frac{3}{4}xy^4\right]_{y=0}^{y=3} dx$
$= \int_1^2 \left(18 - \frac{243}{4}x\right) dx = \left[18x - \frac{243}{8}x^2\right]_1^2 = -\frac{585}{8}$

18. $\int_2^3\int_{-1}^0 \left(xy^2 + yx^{-1}\right) dy\, dx$
$= \int_2^3 \left[\frac{1}{3}xy^3 + \frac{1}{2}y^2x^{-1}\right]_{y=-1}^{y=0} dx = \int_2^3 \left(\frac{1}{3}x - \frac{1}{2}x^{-1}\right) dx$
$= \left[\frac{1}{6}x^2 - \frac{1}{2}\ln x\right]_2^3 = \frac{5}{6} + \ln\sqrt{\frac{2}{3}}$

19. $\int_0^{\pi/6}\int_1^4 x \sin y\, dx\, dy = \left(\int_0^{\pi/6} \sin y\, dy\right)\left(\int_1^4 x\, dx\right)$
$= \left(1 - \frac{\sqrt{3}}{2}\right)\frac{15}{2} = \frac{15\left(2-\sqrt{3}\right)}{4}$

20. $\left[\int_0^1 \frac{1}{1+y}\, dy\right]\left[\int_{-1}^2 (1+x)\, dx\right]$
$= \left[\ln(1+y)\right]_0^1 \left[x + \frac{1}{2}x^2\right]_{-1}^2$
$= (\ln 2)\left(2 + 2 + 1 - \frac{1}{2}\right) = \frac{9}{2}\ln 2$

21. $\iint_R xye^y\, dA = \int_0^2\int_0^1 xye^y\, dy\, dx = \int_0^2 x\, dx \int_0^1 ye^y\, dy$
$= \left[\frac{1}{2}x^2\right]_0^2 \left[e^y(y-1)\right]_0^1$ (by parts)
$= \frac{1}{2}(4-0)\left(0 + e^0\right) = 2$

22. $\int_0^1\int_0^1 xe^{xy}\, dy\, dx = \int_0^1 \left[e^{xy}\right]_{y=0}^{y=1} dx = \int_0^1 (e^x - 1)\, dx$
$= \left[e^x - x\right]_0^1 = e - 2$

23. $\int_0^1\int_1^2 \frac{1}{x+y}\, dx\, dy = \int_0^1 \left[\ln(x+y)\right]_{x=1}^{x=2} dy$
$= \int_0^1 \left[\ln(2+y) - \ln(1+y)\right] dy$
$= \Big[\left[(2+y)\ln(2+y) - (2+y)\right]$
$- \left[(1+y)\ln(1+y) - (1+y)\right]\Big]_0^1$
[by parts separately for each term or by the Table of Integrals]
$= (3\ln 3) - 3 - (2\ln 2) + 2 - \left[(2\ln 2 - 2) - (0 - 1)\right]$
$= 3\ln 3 - 4\ln 2 = \ln\frac{27}{16}$

24. $V = \iint_R (2x + 5y + 1)\, dA = \int_1^4\int_{-1}^0 (2x + 5y + 1)\, dx\, dy$
$= \int_1^4 \left[x^2 + 5xy + x\right]_{x=-1}^{x=0} dy = \int_1^4 5y\, dy = \left.\frac{5}{2}y^2\right]_1^4$
$= \frac{75}{2}$

25. $V = \iint_R \left(x^2 + y^2\right) dA = \int_{-3}^3\int_{-2}^2 \left(x^2 + y^2\right) dx\, dy$
$= \int_{-3}^3 \left[\frac{1}{3}x^3 + y^2x\right]_{x=-2}^{x=2} dy = \int_{-3}^3 \left[\frac{16}{3} + 4y^2\right] dy$
$= \left[\frac{16}{3}y + \frac{4}{3}y^3\right]_{-3}^3 = 2(16 + 36) = 104$

26. $V = \int_1^3\int_{-1}^1 \left(y^2 - x^2\right) dx\, dy = 2\int_1^3\int_0^1 \left(y^2 - x^2\right) dx\, dy$
$= 2\int_1^3 \left[y^2x - \frac{1}{3}x^3\right]_{x=0}^{x=1} dy = 2\int_1^3 \left(y^2 - \frac{1}{3}\right) dy$
$= \frac{2}{3}\left[y^3 - y\right]_1^3 = 16$

27. $A(R) = \frac{\pi}{2}\cdot 1 = \frac{\pi}{2}$, so
$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\, dA$
$= \frac{1}{\pi/2}\int_0^{\pi/2}\int_0^1 x \sin xy\, dy\, dx$
$= \frac{2}{\pi}\int_0^{\pi/2} \left[-\cos xy\right]_{y=0}^{y=1} dx$
$= \frac{2}{\pi}\int_0^{\pi/2} (1 - \cos x)\, dx$
$= \frac{2}{\pi}\left[x - \sin x\right]_0^{\pi/2} = 1 - \frac{2}{\pi}$