

12.2

DOUBLE INTEGRALS OVER GENERAL REGIONS

A Click here for answers.

1–6 ■ Evaluate the iterated integral.

1. $\int_0^1 \int_0^y x \, dx \, dy$
2. $\int_0^1 \int_0^y y \, dx \, dy$
3. $\int_0^2 \int_{\sqrt{x}}^3 (x^2 + y) \, dy \, dx$
4. $\int_0^1 \int_{1-x}^{1+x} (2x - 3y^2) \, dy \, dx$
5. $\int_0^1 \int_0^x \sin(x^2) \, dy \, dx$
6. $\int_0^1 \int_{x-1}^0 \frac{2y}{x+1} \, dy \, dx$

7–19 ■ Evaluate the double integral.

7. $\iint_D xy \, dA$, $D = \{(x, y) \mid 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}$
8. $\iint_D (x - 2y) \, dA$,
 $D = \{(x, y) \mid 1 \leq x \leq 3, 1 + x \leq y \leq 2x\}$
9. $\iint_D (x^2 - 2xy) \, dA$,
 $D = \{(x, y) \mid 0 \leq x \leq 1, \sqrt{x} \leq y \leq 2 - x\}$
10. $\iint_D x \sin y \, dA$,
 $D = \{(x, y) \mid 0 \leq y \leq \pi/2, 0 \leq x \leq \cos y\}$
11. $\iint_D \frac{1}{x} \, dA$, $D = \{(x, y) \mid 1 \leq y \leq e, y^2 \leq x \leq y^4\}$
12. $\iint_D (3x + y) \, dA$,
 $D = \{(x, y) \mid \pi/6 \leq x \leq \pi/4, \sin x \leq y \leq \cos x\}$
13. $\iint_D (y - xy^2) \, dA$,
 $D = \{(x, y) \mid 0 \leq y \leq 1, -y \leq x \leq 1 + y\}$
14. $\iint_D (x^2 + y) \, dA$, D is bounded by $y = x^2$, $y = 2 - x^2$
15. $\iint_D 3xy \, dA$, D is bounded by $y = x$, $y = x^2 - 4x + 4$

S Click here for solutions.

16. $\iint_D e^{x+y} \, dA$, D is bounded by $y = 0$, $y = x$, $x = 1$
17. $\iint_D xy \, dA$, D is the first-quadrant part of the disk with center $(0, 0)$ and radius 1
18. $\iint_D (y^2 - x) \, dA$, D is bounded by $x = y^2$, $x = 3 - 2y^2$
19. $\iint_D ye^x \, dA$,
 D is the triangular region with vertices $(0, 0)$, $(2, 4)$, and $(6, 0)$

20–26 ■ Find the volume of the given solid.

20. Under the paraboloid $z = x^2 + y^2$ and above the region bounded by $y = x^2$ and $x = y^2$
21. Under the paraboloid $z = 3x^2 + y^2$ and above the region bounded by $y = x$ and $x = y^2 - y$
22. Bounded by the paraboloid $z = x^2 + y^2 + 4$ and the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$
23. Bounded by the cylinder $x^2 + z^2 = 9$ and the planes $x = 0$, $y = 0$, $z = 0$, $x + 2y = 2$ in the first octant
24. Bounded by the planes $y = 0$, $z = 0$, $y = x$, and $6x + 2y + 3z = 6$
25. Under the surface $z = 1 + xy$ and above the triangle with vertices $(1, 1)$, $(4, 1)$, and $(3, 2)$
26. Bounded by the cylinder $y^2 + z^2 = 9$ and the planes $y = 3x$, $y = 0$, $z = 0$ in the first octant

27–30 ■ Sketch the region of integration and change the order of integration.

27. $\int_0^1 \int_0^x f(x, y) \, dy \, dx$
28. $\int_0^{\pi/2} \int_0^{\sin x} f(x, y) \, dy \, dx$
29. $\int_0^1 \int_{y^2}^{2-y} f(x, y) \, dx \, dy$
30. $\int_0^4 \int_{y/2}^2 f(x, y) \, dx \, dy$

12.2 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{1}{6}$

2. $\frac{1}{3}$

3. $16 \left(1 - \frac{\sqrt{2}}{7} \right)$

4. $-\frac{13}{6}$

5. $\frac{1}{2} (1 - \cos 1)$

6. $\frac{5}{2} - 4 \ln 2$

7. $\frac{1}{12}$

8. $-\frac{34}{3}$

9. $-\frac{19}{42}$

10. $\frac{1}{6}$

11. 2

12. $\frac{3\sqrt{2}-1-\sqrt{3}}{4}\pi + \frac{14-13\sqrt{3}}{8}$

13. $\frac{3}{4}$

14. $\frac{16}{5}$

15. $\frac{2613}{40}$

16. $\frac{1}{2} (e^2 - 2e + 1)$

17. $\frac{1}{8}$

18. $-\frac{24}{5}$

19. $e^6 - 9e^2 - 4$

20. $\frac{6}{35}$

21. $\frac{144}{35}$

22. $\frac{13}{6}$

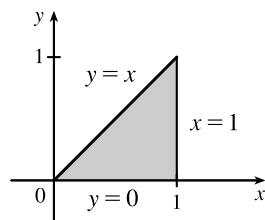
23. $\frac{1}{6} (11\sqrt{5} - 27) + \frac{9}{2} \sin^{-1} \left(\frac{2}{3} \right)$

24. $\frac{1}{4}$

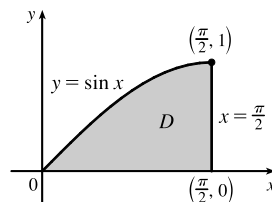
25. $\frac{55}{8}$

26. 3

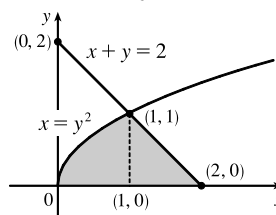
27. $\int_0^1 \int_y^1 f(x, y) dx dy$



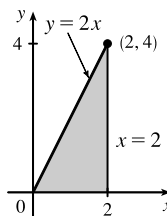
28. $\int_0^1 \int_{\sin^{-1} y}^{\pi/2} f(x, y) dx dy$



29. $\int_0^1 \int_0^{\sqrt{x}} f(x, y) dy dx + \int_1^2 \int_0^{2-x} f(x, y) dy dx$



30. $\int_0^2 \int_0^{2x} f(x, y) dy dx$



12.2 SOLUTIONS

 Click here for exercises.

$$1. \int_0^1 \int_0^y x \, dx \, dy = \int_0^1 \left[\frac{1}{2} x^2 \right]_0^y dy = \int_0^1 \left[\frac{1}{2} y^2 \right] dy = \frac{1}{6}$$

$$2. \int_0^1 \int_0^y y \, dx \, dy = \int_0^1 y^2 dy = \frac{1}{3}$$

$$\begin{aligned} 3. \int_0^2 \int_{\sqrt{x}}^3 (x^2 + y) \, dy \, dx &= \int_0^2 \left[x^2 y + \frac{1}{2} y^2 \right]_{\sqrt{x}}^3 dx \\ &= \int_0^2 \left[3x^2 + \frac{9}{2} - x^{5/2} - \frac{1}{2} x \right] dx \\ &= \left[x^3 + \frac{9}{2} x - \frac{2}{7} x^{7/2} - \frac{1}{4} x^2 \right]_0^2 = 16 \left(1 - \frac{\sqrt{2}}{7} \right) \end{aligned}$$

$$\begin{aligned} 4. \int_0^1 \int_{1-x}^{1+x} (2x - 3y^2) \, dy \, dx &= \int_0^1 \left[2xy - y^3 \right]_{1-x}^{1+x} dx \\ &= \int_0^1 [4x^2 - (1+x)^3 + (1-x)^3] dx \\ &= \left[\frac{4}{3} x^3 - \frac{1}{4} (1+x)^4 - \frac{1}{4} (1-x)^4 \right]_0^1 \\ &= \frac{4}{3} - 4 + \frac{1}{4} + \frac{1}{4} = -\frac{13}{6} \end{aligned}$$

$$\begin{aligned} 5. \int_0^1 \int_0^x \sin(x^2) \, dy \, dx &= \int_0^1 x \sin(x^2) \, dx \\ &= \frac{1}{2} [-\cos(x^2)]_0^1 = \frac{1}{2} (1 - \cos 1) \end{aligned}$$

$$\begin{aligned} 6. \int_0^1 \int_{x-1}^0 \frac{2y}{x+1} \, dy \, dx &= \int_0^1 \left[\frac{y^2}{x+1} \right]_{x-1}^0 dx \\ &= - \int_0^1 \frac{(1-x)^2}{x+1} dx = - \int_0^1 \left(x - 3 + \frac{4}{x+1} \right) dx \\ &= - \left[\frac{1}{2} x^2 - 3x + 4 \ln(x+1) \right]_0^1 = \frac{5}{2} - 4 \ln 2 \end{aligned}$$

$$\begin{aligned} 7. \int_0^1 \int_{x^2}^{\sqrt{x}} xy \, dy \, dx &= \int_0^1 \left[\frac{1}{2} xy^2 \right]_{x^2}^{\sqrt{x}} dx \\ &= \frac{1}{2} \int_0^1 (x^2 - x^5) dx = \frac{1}{2} \left[\frac{1}{3} x^3 - \frac{1}{6} x^6 \right]_0^1 = \frac{1}{12} \end{aligned}$$

$$\begin{aligned} 8. \int_1^3 \int_{1+x}^{2x} (x - 2y) \, dy \, dx &= \int_1^3 \left[xy - y^2 \right]_{1+x}^{2x} dx \\ &= \int_1^3 [(1+x)^2 - 3x^2 - x] dx \\ &= \left[\frac{1}{3} (1+x)^3 - x^3 - \frac{1}{2} x^2 \right]_1^3 = -\frac{34}{3} \end{aligned}$$

$$\begin{aligned} 9. \int_0^1 \int_{\sqrt{x}}^{2-x} (x^2 - 2xy) \, dy \, dx &= \int_0^1 \left[x^2 y - xy^2 \right]_{\sqrt{x}}^{2-x} dx \\ &= \int_0^1 (-2x^3 + 7x^2 - 4x - x^{5/2}) dx \\ &= \left[-\frac{1}{2} x^4 + \frac{7}{3} x^3 - 2x^2 - \frac{2}{7} x^{7/2} \right]_0^1 = -\frac{19}{42} \end{aligned}$$

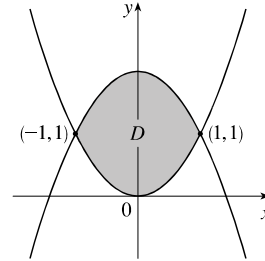
$$\begin{aligned} 10. \int_0^{\pi/2} \int_0^{\cos y} x \sin y \, dx \, dy &= \int_0^{\pi/2} \frac{1}{2} (\cos^2 y \sin y) dy \\ &= -\frac{1}{6} \cos^3 y \Big|_0^{\pi/2} = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} 11. \int_1^e \int_{y^2}^4 (1/x) \, dx \, dy &= \int_1^e (\ln y^4 - \ln y^2) dy \\ &= \int_1^e 2 \ln y \, dy = 2 [y \ln y - y]_1^e = 2 \end{aligned}$$

$$\begin{aligned} 12. \int_{\pi/6}^{\pi/4} \int_{\sin x}^{\cos x} (3x + y) \, dy \, dx &= \int_{\pi/6}^{\pi/4} \left[3xy + \frac{1}{2} y^2 \right]_{y=\sin x}^{y=\cos x} dx \\ &= \int_{\pi/6}^{\pi/4} \left[3x(\cos x - \sin x) + \frac{1}{2} \cos^2 x - \frac{1}{2} \sin^2 x \right] dx \\ &= 3x(\sin x + \cos x) \Big|_{\pi/6}^{\pi/4} - 3 \int_{\pi/6}^{\pi/4} (\sin x + \cos x) dx \\ &\quad + \left[\frac{1}{4} \sin 2x \right]_{\pi/6}^{\pi/4} \\ &= 3 \left(\frac{\pi}{4} \right) \sqrt{2} - \frac{\pi}{2} \cdot \frac{1+\sqrt{3}}{2} + 3 \left[0 + \frac{1-\sqrt{3}}{2} \right] + \frac{1}{4} \left(1 - \frac{\sqrt{3}}{2} \right) \\ &= \frac{3\sqrt{2}-1-\sqrt{3}}{4} \pi + \frac{14-13\sqrt{3}}{8} \end{aligned}$$

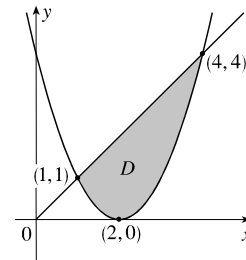
$$\begin{aligned} 13. \int_0^1 \int_{-y}^{1+y} (y - xy^2) \, dx \, dy &= \int_0^1 \left[xy - \frac{1}{2} x^2 y^2 \right]_{x=-y}^{x=1+y} dy \\ &= \int_0^1 \left(-y^3 + \frac{3}{2} y^2 + y \right) dy = -\frac{1}{4} y^4 + \frac{1}{2} y^3 + \frac{1}{2} y^2 \Big|_0^1 = \frac{3}{4} \end{aligned}$$

14.



$$\begin{aligned} \int_{-1}^1 \int_{x^2}^{2-x^2} (x^2 + y) \, dy \, dx &= 2 \int_0^1 \int_{x^2}^{2-x^2} (x^2 + y) \, dy \, dx \\ &= 2 \int_0^1 \left[x^2 y + \frac{1}{2} y^2 \right]_{y=x^2}^{y=2-x^2} dx = 2 \int_0^1 (-2x^4 + 2) dx \\ &= 4 \left[-\frac{1}{5} x^5 + x \right]_0^1 = \frac{16}{5} \end{aligned}$$

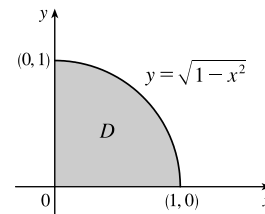
15.



$$\begin{aligned} \int_1^4 \int_{x^2-4x+4}^x 3xy \, dy \, dx &= \int_1^4 \left[\frac{3}{2} xy^2 \right]_{y=x^2-4x+4}^{y=x} dx \\ &= \frac{3}{2} \int_1^4 [x^3 - x(x-2)^4] dx \\ &= \frac{3}{2} \int_1^4 [x^3 - (x-2)^5 - 2(x-2)^4] dx \\ &= \frac{3}{2} \left[\frac{1}{4} x^4 - \frac{1}{6} (x-2)^6 - 2(x-2)^5 \right]_1^4 \\ &= \frac{3}{2} \left(64 - \frac{409}{20} \right) = \frac{2613}{40} \end{aligned}$$

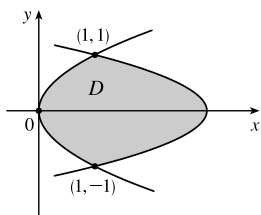
$$\begin{aligned} 16. \int_0^1 \int_0^x e^{x+y} \, dy \, dx &= \int_0^1 (e^{2x} - e^x) dx = \left[\frac{1}{2} e^{2x} - e^x \right]_0^1 \\ &= \frac{1}{2} (e^2 - 2e + 1) \end{aligned}$$

17.



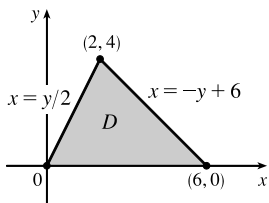
$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx &= \int_0^1 \left[\frac{1}{2} xy^2 \right]_0^{\sqrt{1-x^2}} dx \\ &= \int_0^1 \frac{x - x^3}{2} dx = \frac{1}{2} \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{8} \end{aligned}$$

18.



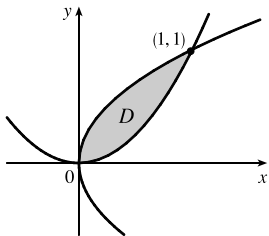
$$\begin{aligned} \int_{-1}^1 \int_{y^2}^{3-2y^2} (y^2 - x) dx dy &= \int_{-1}^1 \left[xy^2 - \frac{1}{2}x^2 \right]_{x=y^2}^{x=3-2y^2} dy \\ &= \int_{-1}^1 \left[(3-2y^2)y^2 - \frac{1}{2}(3-2y^2)^2 - y^2y^2 + \frac{1}{2}(y^2)^2 \right] dy \\ &= \int_{-1}^1 \left(-\frac{9}{2}y^4 + 9y^2 - \frac{9}{2} \right) dy \\ &= \left[-\frac{9}{10}y^5 + 3y^3 - \frac{9}{2}y \right]_{-1}^1 = -\frac{24}{5} \end{aligned}$$

19.



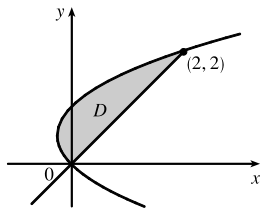
$$\begin{aligned} \int_0^4 \int_{y/2}^{6-y} ye^x dx dy &= \int_0^4 [ye^x]_{x=y/2}^{x=6-y} dy \\ &= \int_0^4 (ye^{6-y} - ye^{y/2}) dy \\ &= \left[y(-e^{6-y} - 2e^{y/2}) \right]_0^4 + \left[-e^{6-y} + 4e^{y/2} \right]_0^4 \\ &\quad \text{(by parts separately for each term)} \\ &= -12e^2 + 3e^2 + e^6 - 4 = e^6 - 9e^2 - 4 \end{aligned}$$

20.



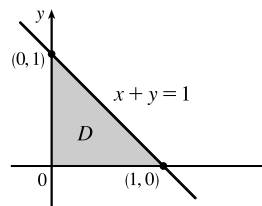
$$\begin{aligned} V &= \int_0^1 \int_{-x^2}^{x^2} (x^2 + y^2) dy dx \\ &= \int_0^1 \left[x^2y + \frac{1}{3}y^3 \right]_{y=-x^2}^{y=x^2} dx \\ &= \int_0^1 \left(x^5/2 - x^4 + \frac{1}{3}x^3/2 - \frac{1}{3}x^6 \right) dx \\ &= \left[\frac{2}{7}x^{7/2} - \frac{1}{5}x^5 + \frac{2}{15}x^{5/2} - \frac{1}{21}x^7 \right]_0^1 = \frac{18}{105} = \frac{6}{35} \end{aligned}$$

21.



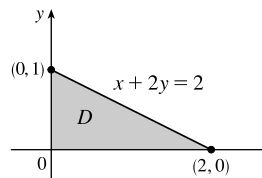
$$\begin{aligned} V &= \int_0^2 \int_{y^2}^{2y} (3x^2 + y^2) dx dy \\ &= \int_0^2 [x^3 + y^2x]_{x=y^2}^{x=2y} dy \\ &= \int_0^2 [2y^3 - (y^6 - 3y^5 + 4y^4 - 2y^3)] dy \\ &= \left[-\frac{1}{7}y^7 + \frac{1}{2}y^6 - \frac{4}{5}y^5 + y^4 \right]_0^2 = \frac{144}{35} \end{aligned}$$

22.



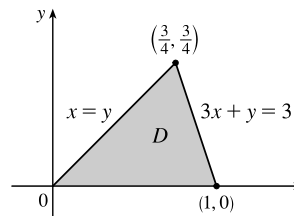
$$\begin{aligned} V &= \int_0^1 \int_0^{1-x} (x^2 + y^2 + 4) dy dx \\ &= \int_0^1 \left[x^2y + \frac{1}{3}y^3 + 4y \right]_{y=0}^{y=1-x} dx \\ &= \int_0^1 \left[x^2 - x^3 + \frac{1}{3}(1-x)^3 + 4(1-x) \right] dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(1-x)^4 - 2(1-x)^2 \right]_0^1 = \frac{13}{6} \end{aligned}$$

23.



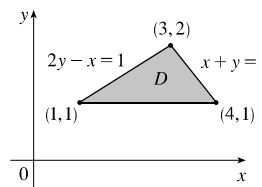
$$\begin{aligned} V &= \int_0^2 \int_0^{1-x/2} \sqrt{9-x^2} dy dx = \int_0^2 [y\sqrt{9-x^2}]_{y=0}^{y=1-x/2} dx \\ &= \int_0^2 \left(\sqrt{9-x^2} - \frac{1}{2}x\sqrt{9-x^2} \right) dx \\ &= \int_0^2 \sqrt{9-x^2} dx + \frac{1}{4} \int_0^2 (-2x\sqrt{9-x^2}) dx \\ &= \left[\frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}(x/3) + \frac{1}{6}(9-x^2)^{3/2} \right]_0^2 \\ &= \sqrt{5} + \frac{9}{2}\sin^{-1}\frac{2}{3} + \frac{5}{6}\sqrt{5} - \frac{1}{6}(27) \\ &= \frac{1}{6}(11\sqrt{5} - 27) + \frac{9}{2}\sin^{-1}\left(\frac{2}{3}\right) \end{aligned}$$

24.



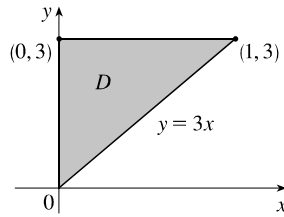
$$\begin{aligned} V &= \int_0^{3/4} \int_y^{(3-y)/3} \frac{1}{3}(6-6x-2y) dx dy \\ &= \int_0^{3/4} \left[\frac{2}{3}(3-y)x - x^2 \right]_{x=y}^{x=(3-y)/3} dy \\ &= \int_0^{3/4} \left[\frac{1}{9}(3-y)^2 - 2y + \frac{5}{9}y^2 \right] dy \\ &= \left[-\frac{1}{27}(3-y)^3 - y^2 + \frac{5}{9}y^3 \right]_0^{3/4} \\ &= -\frac{27}{64} - \frac{9}{16} + \frac{15}{64} + 1 = \frac{1}{4} \end{aligned}$$

25.



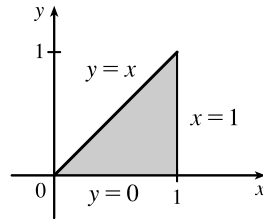
$$\begin{aligned} V &= \int_1^2 \int_{2y-1}^{5-y} (1+xy) dx dy = \int_1^2 \left[x + \frac{1}{2}x^2y \right]_{x=2y-1}^{x=5-y} dy \\ &= \int_1^2 \left(6-3y - \frac{3}{2}y^3 - 3y^2 + 12y \right) dy \\ &= 6y + \frac{9}{2}y^2 - \frac{3}{8}y^4 - y^3 \Big|_1^2 = \frac{55}{8} \end{aligned}$$

26.



$$\begin{aligned} V &= \int_0^3 \int_0^{y/3} \sqrt{9-y^2} \, dx \, dy = \int_0^3 \frac{1}{3} y \sqrt{9-y^2} \, dy \\ &= -\frac{1}{9} (9-y^2)^{3/2} \Big|_0^3 = 3 \end{aligned}$$

27.



Because the region of integration is

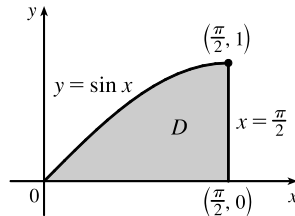
$$D = \{(x, y) \mid 0 \leq y \leq x, 0 \leq x \leq 1\}$$

$$= \{(x, y) \mid y \leq x \leq 1, 0 \leq y \leq 1\}$$

we have

$$\begin{aligned} \int_0^1 \int_0^x f(x, y) \, dy \, dx &= \iint_D f(x, y) \, dA \\ &= \int_0^1 \int_y^1 f(x, y) \, dx \, dy \end{aligned}$$

28.



Because the region of integration is

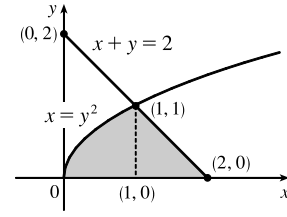
$$D = \{(x, y) \mid 0 \leq y \leq \sin x, 0 \leq x \leq \frac{\pi}{2}\}$$

$$= \{(x, y) \mid \sin^{-1} y \leq x \leq \frac{\pi}{2}, 0 \leq y \leq 1\}$$

we have

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\sin x} f(x, y) \, dy \, dx &= \iint_D f(x, y) \, dA \\ &= \int_0^1 \int_{\sin^{-1} y}^{\pi/2} f(x, y) \, dx \, dy \end{aligned}$$

29.



To reverse the order, we must break the region into two separate type I regions. Because the region of integration is

$$D = \{(x, y) \mid y^2 \leq x \leq 2 - y, 0 \leq y \leq 1\}$$

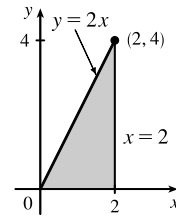
$$= \{(x, y) \mid 0 \leq y \leq \sqrt{x}, 0 \leq x \leq 1\}$$

$$\cup \{0 \leq y \leq 2 - x, 1 \leq x \leq 2\}$$

we have

$$\begin{aligned} \int_0^1 \int_{y^2}^{2-y} f(x, y) \, dx \, dy &= \iint_D f(x, y) \, dA \\ &= \int_0^1 \int_0^{\sqrt{x}} f(x, y) \, dy \, dx + \int_1^2 \int_0^{2-x} f(x, y) \, dy \, dx \end{aligned}$$

30.



Because the region of integration is

$$D = \{(x, y) \mid y/2 \leq x \leq 2, 0 \leq y \leq 4\}$$

$$= \{(x, y) \mid 0 \leq y \leq 2x, 0 \leq x \leq 2\}$$

we have

$$\begin{aligned} \int_0^4 \int_{y/2}^2 f(x, y) \, dx \, dy &= \iint_D f(x, y) \, dA \\ &= \int_0^2 \int_0^{2x} f(x, y) \, dy \, dx \end{aligned}$$