12.4 APPLICATIONS OF DOUBLE INTEGRALS

A Click here for answers.

S Click here for solutions.

1. Electric charge is distributed over the rectangle $0 \leqslant x \leqslant 2$, $1 \leqslant y \leqslant 2$ so that the charge density at (x, y) is $\sigma(x, y) = x^2 + 3y^2$ (measured in coulombs per square meter). Find the total charge on the rectangle.

2. Electric charge is distributed over the unit disk $x^2 + y^2 \leqslant 1$ so that the charge density at (x, y) is+ $\sigma(x, y) = 1 + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.

3–9 ▪ Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

3. $D = \{(x, y) \mid -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$; $\rho(x, y) = x^2$

4. $D = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3\}$; $\rho(x, y) = y$

5. D is the region in the first quadrant bounded by the parabola $y = x^2$ and the line $y = 1$; $\rho(x, y) = xy$

CAS **6.** D is bounded by the parabola $y = 9 - x^2$ and the x-axis; $\rho(x, y) = y$

7. D is bounded by the cardioid $r = 1 + \sin\theta$; $\rho(x, y) = 2$

8. $D = \{(x, y) \mid 0 \leqslant y \leqslant \sin x, 0 \leqslant x \leqslant \pi\}$; $\rho(x, y) = y$

9. $D = \{(x, y) \mid 0 \leqslant y \leqslant \cos x, 0 \leqslant x \leqslant \pi/2\}$; $\rho(x, y) = x$

10. Find the moments of inertia I_x, I_y, I_0 for the lamina of Problem 5.

12.4 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{50}{3}$ C

2. $\frac{3\pi}{2}$ C

3. $\frac{2}{3}$, $\left(0, \frac{1}{2}\right)$

4. 9, $(1, 2)$

5. $\frac{1}{6}$, $\left(\frac{4}{7}, \frac{3}{4}\right)$

6. $\frac{648}{5}$, $\left(0, \frac{36}{7}\right)$

7. 3π, $\left(0, \frac{5}{6}\right)$

8. $\frac{\pi}{4}$, $\left(\frac{\pi}{2}, \frac{16}{9\pi}\right)$

9. $\dfrac{\pi - 2}{2}$, $\left(\dfrac{\pi^2 - 8}{2(\pi - 2)}, \dfrac{\pi}{4}\right)$

10. $\frac{1}{10}$, $\frac{1}{16}$, $\frac{13}{80}$

12.4 SOLUTIONS

E Click here for exercises.

1. $Q = \iint_D \left(x^2 + 3y^2\right) dA = \int_0^2 \int_1^2 \left(x^2 + 3y^2\right) dy\,dx$
$= \int_0^2 \left[x^2 y + y^3\right]_{y=1}^{y=2} dx = \int_0^2 \left(x^2 + 7\right) dx$
$= \left[\frac{1}{3}x^3 + 7x\right]_0^2 = \frac{8}{3} + 14 = \frac{50}{3}$ C

2. $Q = \iint_{0 \le x^2+y^2 \le 1} \left(1 + x^2 + y^2\right) dA$
$= \int_0^{2\pi} \int_0^1 \left(1 + r^2\right) r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 \left(r + r^3\right) dr$
$= 2\pi \left[\frac{1}{2}r^2 + \frac{1}{4}r^4\right]_0^1 = \frac{3\pi}{2}$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_{-1}^1 \int_0^1 x^2\,dy\,dx$
$= \int_{-1}^1 x^2\,dx \int_0^1 dy = \left[\frac{1}{3}x^3\right]_{-1}^1 [y]_0^1 = \frac{2}{3}$
$\overline{x} = \frac{1}{m} \iint_D x\rho(x,y)\,dA = \frac{3}{2} \int_{-1}^1 \int_0^1 x^3\,dy\,dx$
$= \frac{3}{2} \int_{-1}^1 x^3\,dx \int_0^1 dy = \frac{3}{2} \left[\frac{1}{4}x^4\right]_{-1}^1 [y]_0^1 = 0$
$\overline{y} = \frac{1}{m} \iint_D y\rho(x,y)\,dA = \frac{3}{2} \int_{-1}^1 \int_0^1 x^2 y\,dy\,dx$
$= \frac{3}{2} \int_{-1}^1 x^2\,dx \int_0^1 y\,dy = \frac{3}{2} \left[\frac{1}{3}x^3\right]_{-1}^1 \left[\frac{1}{2}y^2\right]_0^1 = \frac{1}{2}$
Hence $(\overline{x}, \overline{y}) = \left(0, \frac{1}{2}\right)$.

4. $m = \int_0^3 \int_0^2 y\,dx\,dy = \int_0^2 dx \int_0^3 y\,dy = 9$,
$M_y = \int_0^3 \int_0^2 xy\,dx\,dy = \int_0^2 x\,dx \int_0^3 y\,dy = 9$ and
$M_x = \int_0^3 \int_0^2 y^2\,dx\,dy = \int_0^2 dx \int_0^3 y^2\,dy = 18$.
Hence $m = 9$, $(\overline{x}, \overline{y}) = (M_y/m, M_x/m) = (1, 2)$.

5. $m = \int_0^1 \int_{x^2}^1 xy\,dy\,dx = \int_0^1 \left(\frac{1}{2}x - \frac{1}{2}x^5\right) dx = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$,
$M_y = \int_0^1 \int_{x^2}^1 x^2 y\,dy\,dx = \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{2}x^6\right) dx =$
$\frac{1}{6} - \frac{1}{14} = \frac{2}{21}$ and $M_x = \int_0^1 \int_{x^2}^1 xy^2\,dy\,dx =$
$\int_0^1 \left(\frac{1}{3}x - \frac{1}{3}x^7\right) dx = \frac{1}{6} - \frac{1}{24} = \frac{1}{8}$.
Hence $m = \frac{1}{6}$, $(\overline{x}, \overline{y}) = \left(\frac{4}{7}, \frac{3}{4}\right)$.

6. $m = \int_{-3}^3 \int_0^{9-x^2} y\,dy\,dx = \int_{-3}^3 \frac{1}{2}\left(81 - 18x^2 + x^4\right) dx$
$= 243 - 162 + \frac{243}{5} = \frac{648}{5} = 3^4 \cdot \frac{8}{5}$
$M_y = 0$ since ρ is independent of x and the region is symmetric about the y-axis, and
$M_x = \int_{-3}^3 \int_0^{9-x^2} y^2\,dy\,dx = \int_{-3}^3 \frac{1}{3}\left(9 - x^2\right)^3\,dx$
$= 2\int_0^3 \left(243 - 81x^2 + 9x^4 - \frac{1}{3}x^6\right) dx$
$= 2\left[3^6 - 3^6 + \frac{1}{5}3^7 - \frac{1}{21}3^7\right]$
$= 2\left[3^6 \cdot \frac{21-5}{35}\right] = 3^6 \cdot \frac{32}{35}$
Hence $m = \frac{648}{5}$, $(\overline{x}, \overline{y}) = \left(0, \frac{36}{7}\right)$.

7. Working in polar coordinates,
$m = \int_0^{2\pi} \int_0^{1+\sin\theta} 2r\,dr\,d\theta = \int_0^{2\pi} (1 + \sin\theta)^2\,d\theta$
$= \int_0^{2\pi} \left[1 + 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right] d\theta$
$= \left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi$
$M_y = 0$ since the lamina is homogeneous and symmetric with respect to the y-axis, and
$M_x = \int_0^{2\pi} \int_0^{1+\sin\theta} \left(2r^2\sin\theta\right) dr\,d\theta$
$= \frac{2}{3}\int_0^{2\pi} (1 + \sin\theta)^3 \sin\theta\,d\theta$
$= \frac{2}{3}\int_0^{2\pi} \left(\sin\theta + 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta\right) d\theta$
$= \frac{2}{3}\left(3\pi + \frac{3}{4}\pi\right) = \frac{5\pi}{2}$
Hence $m = 3\pi$, $(\overline{x}, \overline{y}) = \left(0, \frac{5}{6}\right)$.

8. $m = \int_0^\pi \int_0^{\sin x} y\,dy\,dx = \int_0^\pi \frac{1}{2}\sin^2 x\,dx$
$= \left[\frac{1}{4}x - \frac{1}{8}\sin 2x\right]_0^\pi = \frac{1}{4}\pi$
$M_y = \int_0^\pi \int_0^{\sin x} xy\,dy\,dx = \int_0^\pi \frac{1}{2}x\sin^2 x\,dx$
$= \left[\frac{1}{8}x^2 - \frac{1}{8}x\sin 2x - \frac{1}{16}\cos 2x\right]_0^\pi = \frac{1}{8}\pi^2$
and
$M_x = \int_0^\pi \int_0^{\sin x} y^2\,dy\,dx = \int_0^\pi \frac{1}{3}\sin^3 x\,dx$
$= \frac{1}{3}\left[-\cos x + \frac{1}{3}\cos^3 x\right]_0^\pi = \frac{4}{9}$
Hence $m = \frac{\pi}{4}$, $(\overline{x}, \overline{y}) = \left(\frac{\pi}{2}, \frac{16}{9\pi}\right)$.

9. $m = \int_0^{\pi/2} \int_0^{\cos x} x\,dy\,dx = \int_0^{\pi/2} x\cos x\,dx$
$= [x\sin x + \cos x]_0^{\pi/2} = \frac{\pi}{2} - 1$
$M_y = \int_0^{\pi/2} \int_0^{\cos x} x^2\,dy\,dx = \int_0^{\pi/2} x^2\cos x\,dx$
$= \left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\pi/2} = \frac{\pi^2}{4} - 2$
and
$M_x = \int_0^{\pi/2} \int_0^{\cos x} xy\,dy\,dx = \int_0^{\pi/2} \frac{1}{2}x\cos^2 x\,dx$
$= \frac{1}{2}\left[x^2 - x + (x-1)\sin x\cos x\right]_0^{\pi/2} = \frac{\pi^2}{8} - \frac{\pi}{4}$
Hence $m = \dfrac{\pi - 2}{2}$, $(\overline{x}, \overline{y}) = \left(\dfrac{\pi^2 - 8}{2(\pi - 2)}, \dfrac{\pi}{4}\right)$.

10. $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^1 \int_{x^2}^1 y^2(xy)\,dy\,dx$
$= \int_0^1 \left[\frac{1}{4}xy^4\right]_{y=x^2}^{y=1} dx = \int_0^1 \frac{1}{4}\left(x - x^9\right) dx$
$= \frac{1}{8} - \frac{1}{40} = \frac{1}{10}$
$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^1 \int_{x^2}^1 x^3 y\,dy\,dx$
$= \int_0^1 \left[\frac{1}{2}x^3 y^2\right]_{y=x^2}^{y=1} dx = \int_0^1 \frac{1}{2}\left(x^3 - x^7\right) dx$
$= \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$
$I_0 = I_x + I_y = \frac{13}{80}$.