12.6 TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

A Click here for answers.

S Click here for solutions.

1–2 ▪ Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1. $(3, \pi/2, 1)$

2. $(\sqrt{2}, \pi/4, \sqrt{2})$

3–8 ▪ Change from rectangular to cylindrical coordinates.

3. $(-1, 0, 0)$

4. $(1, 1, 1)$

5. $(\sqrt{3}, 1, 4)$

6. $(-\sqrt{2}, \sqrt{2}, 0)$

7. $(4, 4, 4)$

8. $(-1, \sqrt{3}, 2)$

9–12 ▪ Write the equation in cylindrical coordinates.

9. $x^2 + y^2 + z^2 = 16$

10. $x^2 + y^2 - z^2 = 16$

11. $x + 2y + 3z = 6$

12. $x^2 + y^2 = 2z$

13–14 ▪ Sketch the solid whose volume is given by the integral and evaluate the integral.

13. $\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\,dz\,dr\,d\theta$

14. $\int_1^3 \int_0^{\pi/2} \int_r^3 r\,dz\,d\theta\,dr$

15. Evaluate $\iiint_E (x^2 + y^2)\,dV$, where E is the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = -1$ and $z = 2$.

16. Evaluate $\iiint_E \sqrt{x^2 + y^2}\,dV$, where E is the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

17. Evaluate $\iiint_E y\,dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, above the xy-plane, and below the plane $z = x + 2$.

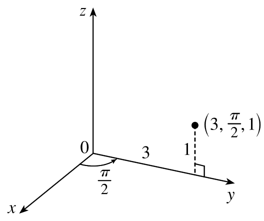
18. Evaluate $\iiint_E xz\,dV$, where E is bounded by the planes $z = 0$, $z = y$, and the cylinder $x^2 + y^2 = 1$ in the half-space $y \geqslant 0$.

12.6 ANSWERS

E Click here for exercises.

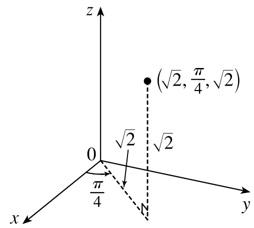
S Click here for solutions.

1.



$(0, 3, 1)$

2.



$(1, 1, \sqrt{2})$

3. $(1, \pi, 0)$

4. $\left(\sqrt{2}, \frac{\pi}{4}, 1\right)$

5. $\left(2, \frac{\pi}{6}, 4\right)$

6. $\left(2, \frac{3\pi}{4}, 0\right)$

7. $\left(4\sqrt{2}, \frac{\pi}{4}, 4\right)$

8. $\left(2, \frac{2\pi}{3}, 2\right)$

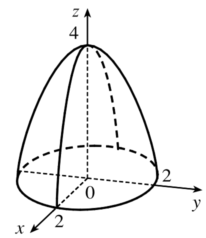
9. $r^2 + z^2 = 16$

10. $r^2 - z^2 = 16$

11. $r\cos\theta + 2r\sin\theta + 3z = 6$

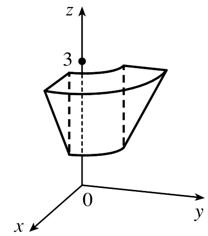
12. $r^2 = 2z$

13.



8π

14.



$\frac{5\pi}{3}$

15. 24π

16. $\frac{324\pi}{5}$

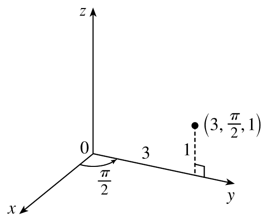
17. 0

18. 0

12.6 SOLUTIONS

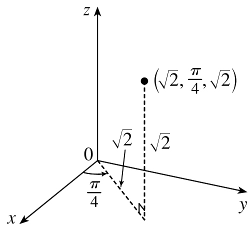
E Click here for exercises.

1.



$x = 3\cos\frac{\pi}{2} = 0$,
$y = 3\sin\frac{\pi}{2} = 3$, and $z = 1$,
so the point is $(0, 3, 1)$ in
rectangular coordinates.

2.



$x = \sqrt{2}\cos\frac{\pi}{4} = 1$,
$y = \sqrt{2}\sin\frac{\pi}{4} = 1$, $z = \sqrt{2}$,
so the point is $(1, 1, \sqrt{2})$ in
rectangular coordinates.

3. $r^2 = (-1)^2 + (0)^2 = 1$ so $r = 1$; $z = 0$; $\tan\theta = 0$ so $\theta = 0$ or π. But $x = -1$ so $\theta = \pi$ and the point is $(1, \pi, 0)$.

4. $r^2 = 1^2 + 1^2 = 2$ or $r = \sqrt{2}$, $\tan\theta = \frac{1}{1}$ so $\theta = \frac{\pi}{4}$ and $z = 1$. Thus in cylindrical coordinates the point is $\left(\sqrt{2}, \frac{\pi}{4}, 1\right)$.

5. $r^2 = 4$ so $r = 2$, $\tan\theta = \dfrac{1}{\sqrt{3}}$ so $\theta = \dfrac{\pi}{6}$ and $z = 4$. Thus the point in cylindrical coordinates is $\left(2, \frac{\pi}{6}, 4\right)$.

6. $r^2 = 4$ so $r = 2$; $\tan\theta = \sqrt{2}/\left(-\sqrt{2}\right) = -1$ and the point $\left(-\sqrt{2}, \sqrt{2}\right)$ is in the second quadrant of the xy-plane so $\theta = \frac{3\pi}{4}$; $z = 0$. The point is $\left(2, \frac{3\pi}{4}, 0\right)$.

7. $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$; $z = 4$; $\tan\theta = \frac{4}{4}$, so $\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$, but both x and y are positive, so $\theta = \frac{\pi}{4}$ and the point is $\left(4\sqrt{2}, \frac{\pi}{4}, 4\right)$.

8. $r = \sqrt{1+3} = 2$; $\tan\theta = -\dfrac{\sqrt{3}}{1}$, so $\theta = \frac{2\pi}{3}$ or $\theta = \frac{5\pi}{3}$, but x is negative and y is positive, so $\theta = \frac{2\pi}{3}$ and the point is $\left(2, \frac{2\pi}{3}, 2\right)$.

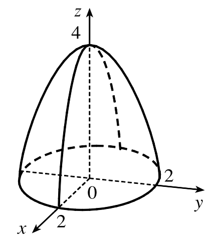
9. $r^2 = x^2 + y^2$, so $r^2 + z^2 = 16$.

10. $r^2 - z^2 = 16$

11. $r\cos\theta + 2r\sin\theta + 3z = 6$

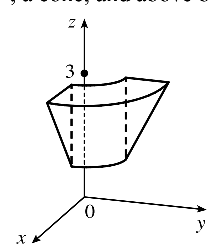
12. $r^2 = 2z$

13. The region of integration is given in cylindrical coordinates by $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, 0 \le z \le 4 - r^2\right\}$. This represents the solid region bounded above by $z = 4 - r^2 = 4 - x^2 - y^2$, a paraboloid, and below by the xy-plane.



$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left(4r - r^3\right) dr\,d\theta$$
$$= \int_0^{2\pi}\left[2r^2 - \tfrac{1}{4}r^4\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi}(8-4)\,d\theta = 4\theta\Big]_0^{2\pi} = 8\pi$$

14. The region of integration is given in cylindrical coordinates by $E = \left\{(r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{2}, 1 \le r \le 3, r \le z \le 3\right\}$. This represents the solid in the first octant between the cylinders $r = 1$ and $r = 3$ and bounded below by $z = r = \sqrt{x^2 + y^2}$, a cone, and above by the plane $z = 3$.



$$\int_1^3\int_0^{\pi/2}\int_r^3 r\,dz\,d\theta\,dr = \int_1^3\int_0^{\pi/2}\left(3r - r^2\right) d\theta\,dr$$
$$= \int_1^3 \tfrac{\pi}{2}\left(3r - r^2\right) dr = \tfrac{\pi}{2}\left[\tfrac{3}{2}r^2 - \tfrac{1}{3}r^3\right]_1^3$$
$$= \tfrac{\pi}{2}\left(\tfrac{27}{2} - \tfrac{27}{3} - \tfrac{3}{2} + \tfrac{1}{3}\right) = \tfrac{5\pi}{3}$$

15. $\iiint_E (x^2+y^2)\,dV = \int_{-1}^{2}\int_0^{2\pi}\int_0^2 (r^2)\,r\,dr\,d\theta\,dz$

$= (3)(2\pi)\left[\frac{1}{4}r^4\right]_0^2 = 24\pi$

16. $\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2\,dz\,dr\,d\theta$

$= 2\pi\int_0^3 (9r^2 - r^4)\,dr = 2\pi\left(81-\frac{243}{5}\right) = \frac{324\pi}{5}$

17. In cylindrical coordinates E is bounded by the cylinders $r=1$ and $r=2$, the plane $z = x+2 = r\cos\theta + 2$, and the xy-plane, so E is given by $\{(r,\theta,z) \mid 0\le\theta\le 2\pi, 1\le r\le 2, 0\le z\le r\cos\theta+2\}$. Thus

$$\begin{aligned}\iiint_E y\,dV &= \int_0^{2\pi}\int_1^2\int_0^{2+r\cos\theta}(r\sin\theta)\,r\,dz\,dr\,d\theta\\ &= \int_0^{2\pi}\int_1^2 r^2\sin\theta\,[z]_{z=0}^{z=2+r\cos\theta}\,dr\,d\theta\\ &= \int_0^{2\pi}\int_1^2 (2r^2+r^3\cos\theta)\sin\theta\,dr\,d\theta\\ &= \int_0^{2\pi}\left[\tfrac{2}{3}r^3+\tfrac{1}{4}r^4\cos\theta\right]_{r=1}^{r=2}\sin\theta\,d\theta\\ &= \int_0^{2\pi}\left(\tfrac{14}{3}+\tfrac{15}{4}\cos\theta\right)\sin\theta\,d\theta\\ &= \left[-\tfrac{14}{3}\cos\theta-\tfrac{15}{8}\cos^2\theta\right]_0^{2\pi} = 0\end{aligned}$$

18. In cylindrical coordinates, E is bounded by the cylinder $r=1$ and the planes $z=0$, $z=y=r\sin\theta$ with $y\ge 0 \quad\Rightarrow\quad 0\le\theta\le\pi$, so E is given by $\{(r,\theta,z)\mid 0\le\theta\le\pi, 0\le r\le 1, 0\le z\le r\sin\theta\}$. Thus

$$\begin{aligned}\iiint_E xz\,dV &= \int_0^{\pi}\int_0^1\int_0^{r\sin\theta} r^2 z\cos\theta\,dz\,dr\,d\theta\\ &= \int_0^{\pi}\int_0^1\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=r\sin\theta} r^2\cos\theta\,dr\,d\theta\\ &= \tfrac{1}{2}\int_0^{\pi}\int_0^1 r^4\sin^2\theta\cos\theta\,dr\,d\theta\\ &= \tfrac{1}{2}\int_0^{\pi}\left[\tfrac{1}{5}r^5\right]_{r=0}^{r=1}\sin^2\theta\cos\theta\,d\theta\\ &= \tfrac{1}{10}\int_0^{\pi}(\sin^2\theta\cos\theta)\,d\theta = \tfrac{1}{30}\sin^3\theta\Big]_0^{\pi} = 0\end{aligned}$$