12.7 TRIPLE INTEGRALS IN SPHERICAL COORDINATES

A Click here for answers.

S Click here for solutions.

1–3 ▪ Change from spherical to rectangular coordinates.

1. $(2, \pi/2, 3\pi/4)$

2. $(4, \pi/4, \pi/6)$

3. $(2, \pi/4, \pi/4)$

4–7 ▪ Change from rectangular to spherical coordinates.

4. $(-3, 0, 0)$

5. $(1, 1, \sqrt{2})$

6. $(\sqrt{3}, 0, 1)$

7. $(-\sqrt{3}, -3, -2)$

8–11 ▪ Change from cylindrical to spherical coordinates.

8. $(\sqrt{2}, \pi/4, 0)$

9. $(1, \pi/2, 1)$

10. $(4, \pi/3, 4)$

11. $(12, \pi, 5)$

12–15 ▪ Write the equation in spherical coordinates.

12. $x^2 + y^2 + z^2 = 16$

13. $x^2 + y^2 - z^2 = 16$

14. $x + 2y + 3z = 6$

15. $x^2 + y^2 = 2z$

16–17 ▪ Sketch the solid whose volume is given by the integral and evaluate the integral.

16. $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

17. $\int_0^{\pi/3} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

18–21 ▪ Use spherical coordinates.

18. Evaluate $\iiint_E xe^{(x^2+y^2+z^2)^2} \, dV$, where E is the solid that lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

19. Evaluate $\iiint_E \sqrt{x^2 + y^2 + z^2} \, dV$, where E is bounded below by the cone $\phi = \pi/6$ and above by the sphere $\rho = 2$.

20. Evaluate $\iiint_E x^2 \, dV$, where E lies between the spheres $\rho = 1$ and $\rho = 3$ and above the cone $\phi = \pi/4$.

21. Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4\cos\phi$.

12.7 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $(0, \sqrt{2}, -\sqrt{2})$

2. $(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$

3. $(1, 1, \sqrt{2})$

4. $(3, \pi, \frac{\pi}{2})$

5. $(2, \frac{\pi}{4}, \frac{\pi}{4})$

6. $(2, 0, \frac{\pi}{3})$

7. $(4, \frac{4\pi}{3}, \frac{2\pi}{3})$

8. $(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2})$

9. $(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4})$

10. $(4\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4})$

11. $(13, \pi, \cos^{-1}(\frac{5}{13}))$

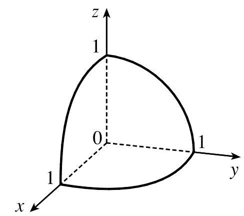
12. $\rho = 4$

13. $\rho^2 (1 - 2\cos^2\phi) = 16$

14. $\rho(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi) = 6$

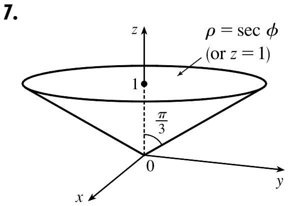
15. $\rho\sin^2\phi = 2\cos\phi$

16.



$\frac{\pi}{6}$

17.



π

18. $\frac{1}{16}\pi(e^{16} - e)$

19. $4\pi(2 - \sqrt{3})$

20. $121\pi\left(\frac{8-5\sqrt{2}}{30}\right)$

21. $(0, 0, 2.1)$

12.7 SOLUTIONS

E Click here for exercises.

1. $x = 2\sin\frac{3\pi}{4}\cos\frac{\pi}{2} = 0$, $y = 2\sin\frac{3\pi}{4}\sin\frac{\pi}{2} = \sqrt{2}$ and $z = 2\cos\frac{3\pi}{4} = -\sqrt{2}$ so the point is $\left(0, \sqrt{2}, -\sqrt{2}\right)$.

2. $x = 4\sin\frac{\pi}{6}\cos\frac{\pi}{4} = 4\left(\frac{1}{2}\right)\frac{1}{\sqrt{2}} = \sqrt{2}$, $y = 4\sin\frac{\pi}{6}\sin\frac{\pi}{4} = \sqrt{2}$ and $z = 4\cos\frac{\pi}{6} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$ so the point is $\left(\sqrt{2}, \sqrt{2}, 2\sqrt{3}\right)$.

3. $x = 2\sin\frac{\pi}{4}\cos\frac{\pi}{4} = 1$, $y = 2\sin\frac{\pi}{4}\sin\frac{\pi}{4} = 1$ and $z = 2\cos\frac{\pi}{4} = \sqrt{2}$ so the point is $\left(1, 1, \sqrt{2}\right)$ in rectangular coordinates.

4. $\rho = \sqrt{9+0+0} = 3$, $\cos\phi = \frac{0}{3} = 0$ so $\phi = \frac{\pi}{2}$, and $\cos\theta = \dfrac{-3}{3\sin\frac{\pi}{2}} = -1$ so $\theta = \pi$, thus spherical coordinates are $\left(3, \pi, \frac{\pi}{2}\right)$.

5. $\rho = \sqrt{1+1+2} = 2$, $\cos\phi = \dfrac{\sqrt{2}}{2}$ so $\phi = \frac{\pi}{4}$, and $\cos\theta = \dfrac{1}{2\sin\frac{\pi}{4}} = \dfrac{1}{\sqrt{2}}$ so $\theta = \frac{\pi}{4}$, thus in spherical coordinates the point is $\left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)$.

6. $\rho = \sqrt{3+1} = 2$, $\cos\phi = \frac{1}{2}$ so $\phi = \frac{\pi}{3}$, and $\cos\theta = \dfrac{\sqrt{3}}{2\sin\frac{\pi}{3}} = \dfrac{\sqrt{3}\cdot 2}{2\cdot\sqrt{3}} = 1$ so $\theta = 0$, thus the point is $\left(2, 0, \frac{\pi}{3}\right)$ in spherical coordinates.
Note: It is also apparent that $\theta = 0$ since the point is in the xz-plane and $x > 0$.

7. $\rho = \sqrt{3+9+4} = 4$, $\cos\phi = -\frac{2}{4} = -\frac{1}{2}$ so $\phi = \frac{2\pi}{3}$, and $\cos\theta = -\dfrac{\sqrt{3}}{4\sin\frac{5\pi}{6}} = -\dfrac{\sqrt{3}\cdot 2}{4\cdot\sqrt{3}} = -\frac{1}{2}$ and $y = -3$ so $\theta = \frac{4\pi}{3}$. Thus in spherical coordinates the point is $\left(4, \frac{4\pi}{3}, \frac{2\pi}{3}\right)$.

8. $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} = \sqrt{2+0} = \sqrt{2}$; $\theta = \frac{\pi}{4}$; $z = \rho\cos\phi = \sqrt{2}\cos\phi = 0$ so $\phi = \frac{\pi}{2}$ and the point is $\left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

9. $\rho = \sqrt{r^2+z^2} = \sqrt{1+1} = \sqrt{2}$, $z = 1 = \sqrt{2}\cos\phi$, so $\phi = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$ and the point is $\left(\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$.

10. $\rho = \sqrt{r^2+z^2} = \sqrt{4^2+4^2} = 4\sqrt{2}$; $\theta = \frac{\pi}{3}$; $z = 4 = 4\sqrt{2}\cos\phi$ so $\cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$ and the point is $\left(4\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$.

11. $\rho = \sqrt{r^2+z^2} = \sqrt{12^2+5^2} = 13$, $z = 5 = 13\cos\phi$, so $\phi = \cos^{-1}\left(\frac{5}{13}\right)$, $\theta = \pi$ and the point is $\left(13, \pi, \cos^{-1}\left(\frac{5}{13}\right)\right)$.

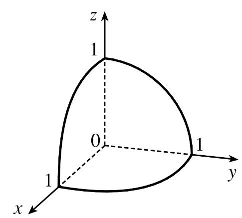
12. $\rho^2 = x^2+y^2+z^2$, so $\rho^2 = 16$ or $\rho = 4$.

13. $x^2+y^2-z^2 = x^2+y^2+z^2-2z^2$, so $\rho^2 - 2\rho^2\cos^2\phi = 16$ or $\rho^2\left(1-2\cos^2\phi\right) = 16$.

14. $\rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi = 6$ or $\rho\left(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi\right) = 6$.

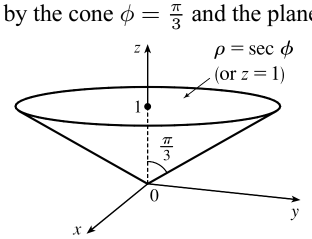
15. $\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = 2\rho\cos\phi$ or $\rho^2\sin^2\phi = 2\rho\cos\phi$ or $\rho\sin^2\phi = 2\cos\phi$.

16. The region of integration is given in spherical coordinates by $E = \left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\right\}$. Thus E is the solid in the first octant bounded by the sphere $\rho = x^2+y^2+z^2 = 1$ and the three coordinate planes.



$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\tfrac{1}{3}\rho^3\sin\phi\right]_{\rho=0}^{\rho=1} d\theta\,d\phi$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\tfrac{1}{3}\sin\phi\,d\theta\,d\phi = \int_0^{\pi/2}\tfrac{1}{3}\sin\phi\,[\theta]_{\theta=0}^{\theta=\pi/2}\,d\phi$$
$$= \tfrac{1}{3}\int_0^{\pi/2}\tfrac{\pi}{2}\sin\phi\,d\phi = \tfrac{\pi}{6}\left[-\cos\phi\right]_0^{\pi/2} = \tfrac{\pi}{6}$$

17. The region of integration is given in spherical coordinates by $E = \left\{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{3}, 0 \le \rho \le \sec\phi\right\}$. Since $\rho = \sec\phi$ is equivalent to $\rho\cos\phi = z = 1$, E is the solid bounded by the cone $\phi = \frac{\pi}{3}$ and the plane $z = 1$.



$$\int_0^{\pi/3}\int_0^{2\pi}\int_0^{\sec\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi/3}\int_0^{2\pi}\left[\tfrac{1}{3}\rho^3\sin\phi\right]_{\rho=0}^{\rho=\sec\phi} d\theta\,d\phi$$
$$= \frac{1}{3}\int_0^{\pi/3}\int_0^{2\pi}\frac{\sin\phi}{\cos^3\phi}\,d\theta\,d\phi$$
$$= \tfrac{2\pi}{3}\int_0^{\pi/3}\left(\tan\phi\sec^2\phi\right)d\phi = \tfrac{2\pi}{3}\left[\tfrac{1}{2}\tan^2\phi\right]_0^{\pi/3} = \pi$$

18. $\iiint_E xe^{(x^2+y^2+z^2)^2}\,dV$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\sin\phi\cos\theta)\, e^{\rho^4}\left(\rho^2\sin\phi\right) d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/2}\cos\theta\, d\theta\, \int_0^{\pi/2}\sin^2\phi\, d\phi\, \int_1^2 \rho^3 e^{\rho^4}\, d\rho$$

$$= [\sin\theta]_0^{\pi/2}\, \left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_0^{\pi/2}\, \left[\tfrac{1}{4}e^{\rho^4}\right]_1^2$$

$$= (1)\left(\tfrac{\pi}{4}\right)\left[\tfrac{1}{4}\left(e^{16}-e\right)\right] = \tfrac{1}{16}\pi\left(e^{16}-e\right)$$

19. $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$

$$= \int_0^{2\pi}\int_0^{\pi/6}\int_0^2 (\rho)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta\, \int_0^{\pi/6}\sin\phi\, d\phi\, \int_0^2 \rho^3\, d\rho$$

$$= [\theta]_0^{2\pi}\, [-\cos\phi]_0^{\pi/6}\, \left[\tfrac{1}{4}\rho^4\right]_0^2 = (2\pi)\left(1-\tfrac{\sqrt{3}}{2}\right)(4)$$

$$= 8\pi\left(1-\tfrac{\sqrt{3}}{2}\right) = 4\pi\left(2-\sqrt{3}\right)$$

20. $\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_1^3 \rho^4\sin^3\phi\cos^2\theta\, d\rho\, d\phi\, d\theta$

$$= \int_0^{2\pi}\cos^2\theta\, d\theta\left(\int_0^{\pi/4}\sin^3\phi\, d\phi\right)\int_1^3 \rho^4\, d\rho$$

$$= \pi\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/4}\left[\tfrac{1}{5}\rho^5\right]_1^3$$

$$= \pi\left(\tfrac{8-5\sqrt{2}}{12}\right)\left(\tfrac{242}{5}\right) = 121\pi\left(\tfrac{8-5\sqrt{2}}{30}\right)$$

21. By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^3\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\cos\phi\sin\phi\left(64\cos^4\phi\right) d\phi\, d\theta$$

$$= \int_0^{2\pi} 64\left[-\tfrac{1}{6}\cos^6\phi\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi}\tfrac{21}{2}\, d\theta = 21\pi$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = (0, 0, 2.1)$.