

13.3 THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

A Click here for answers.

1–9 ■ Determine whether or not \mathbf{F} is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

1. $\mathbf{F}(x, y) = (2x - 3y)\mathbf{i} + (2y - 3x)\mathbf{j}$
2. $\mathbf{F}(x, y) = (3x^2 - 4y)\mathbf{i} + (4y^2 - 2x)\mathbf{j}$
3. $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + x^2\mathbf{j}$
4. $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + (y^2 + x)\mathbf{j}$
5. $\mathbf{F}(x, y) = (1 + 4x^3y^3)\mathbf{i} + 3x^4y^2\mathbf{j}$
6. $\mathbf{F}(x, y) = (y \cos x - \cos y)\mathbf{i} + (\sin x + x \sin y)\mathbf{j}$
7. $\mathbf{F}(x, y) = (e^{2x} + x \sin y)\mathbf{i} + x^2 \cos y\mathbf{j}$
8. $\mathbf{F}(x, y) = (ye^{xy} + 4x^3y)\mathbf{i} + (xe^{xy} + x^4)\mathbf{j}$
9. $\mathbf{F}(x, y) = (x + y^2)\mathbf{i} + (2xy + y^2)\mathbf{j}$

10–17 ■ (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C .

10. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$,
 C is the arc of the parabola $y = x^2$ from $(-1, 1)$ to $(3, 9)$
11. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$,
 C is the arc of the curve $y = x^4 - x^3$ from $(1, 0)$ to $(2, 8)$
12. $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$,
 $C: \mathbf{r}(t) = \sin t\mathbf{i} + (t^2 + 1)\mathbf{j}$, $0 \leq t \leq \pi/2$

S Click here for solutions.

13. $\mathbf{F}(x, y) = e^{2y}\mathbf{i} + (1 + 2xe^{2y})\mathbf{j}$,
 $C: \mathbf{r}(t) = te^t\mathbf{i} + (1 + t)\mathbf{j}$, $0 \leq t \leq 1$
14. $\mathbf{F}(x, y, z) = y\mathbf{i} + (x + z)\mathbf{j} + y\mathbf{k}$,
 C is the line segment from $(2, 1, 4)$ to $(8, 3, -1)$
15. $\mathbf{F}(x, y, z) = 2xy^3z^4\mathbf{i} + 3x^2y^2z^4\mathbf{j} + 4x^2y^3z^3\mathbf{k}$,
 $C: x = t, y = t^2, z = t^3$, $0 \leq t \leq 2$
16. $\mathbf{F}(x, y, z) = (2xz + \sin y)\mathbf{i} + x \cos y\mathbf{j} + x^2\mathbf{k}$,
 $C: \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 2\pi$
17. $\mathbf{F}(x, y, z) = 4xe^z\mathbf{i} + \cos y\mathbf{j} + 2x^2e^z\mathbf{k}$,
 $C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \leq t \leq 1$

18–19 ■ Show that the line integral is independent of path and evaluate the integral.

18. $\int_C 2x \sin y \, dx + (x^2 \cos y - 3y^2) \, dy$,
 C is any path from $(-1, 0)$ to $(5, 1)$
19. $\int_C (2y^2 - 12x^3y^3) \, dx + (4xy - 9x^4y^2) \, dy$,
 C is any path from $(1, 1)$ to $(3, 2)$

20. Find the work done by the force field

$$\mathbf{F}(x, y) = x^2y^3\mathbf{i} + x^3y^2\mathbf{j}$$

in moving an object from $P(0, 0)$ to $Q(2, 1)$.

13.3 **ANSWERS****E** [Click here for exercises.](#)

1. $f(x, y) = x^2 - 3xy + y^2 + K$
2. Not conservative
3. Not conservative
4. $f(x, y) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + K$
5. $f(x, y) = x + x^4y^3 + K$
6. $f(x, y) = y \sin x - x \cos y + K$
7. Not conservative
8. $f(x, y) = e^{xy} + x^4y + K$
9. $f(x, y) = \frac{1}{2}x^2 + xy^2 + \frac{1}{3}y^3 + K$
10. (a) $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$
(b) 44
11. (a) $f(x, y) = xy$
(b) 16
12. (a) $f(x, y) = x^2y^3$
(b) $\frac{1}{64}(\pi^2 + 4)^3$

S [Click here for solutions.](#)

13. (a) $f(x, y) = xe^{2y} + y$
(b) $e^5 + 1$
14. (a) $f(x, y, z) = xy + yz$
(b) 15
15. (a) $f(x, y, z) = x^2y^3z^4$
(b) 2^{20}
16. (a) $f(x, y, z) = x^2z + x \sin y$
(b) 2π
17. (a) $f(x, y, z) = 2x^2e^z + \sin y$
(b) $2e + \sin 1$
18. $25 \sin 1 - 1$
19. -1919
20. $\frac{8}{3}$

13.3 SOLUTIONS



1. $\frac{\partial}{\partial y}(2x - 3y) = -3 = \frac{\partial}{\partial x}(2y - 3x)$ and the domain of \mathbf{F} is \mathbb{R}^2 which is open and simply-connected, so \mathbf{F} is conservative. Thus there exists f such that $\nabla f = \mathbf{F}$, that is, $f_x(x, y) = 2x - 3y$ and $f_y(x, y) = 2y - 3x$. But $f_x(x, y) = 2x - 3y$ implies $f(x, y) = x^2 - 3xy + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x, y) = -3x + g'(y)$. Thus $2y - 3x = -3x + g'(y)$ so $g'(y) = 2y$ and $g(y) = y^2 + K$ where K is a constant. Hence $f(x, y) = x^2 - 3xy + y^2 + K$ is a potential for \mathbf{F} .
2. $\frac{\partial}{\partial y}(3x^2 - 4y) = -4$, $\frac{\partial}{\partial x}(4y^2 - 2x) = -2$ and these are not equal, so \mathbf{F} is not conservative.
3. $\frac{\partial}{\partial y}(x^2 + y) = 1$, $\frac{\partial}{\partial x}(x^2) = 2x$ and these are not equal, so \mathbf{F} is not conservative.
4. $\frac{\partial}{\partial y}(x^2 + y) = 1 = \frac{\partial}{\partial x}(y^2 + x)$ and the domain of \mathbf{F} is \mathbb{R}^2 which is open and simply-connected. Thus \mathbf{F} is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = x^2 + y$ implies $f(x, y) = \frac{1}{3}x^3 + xy + g(y)$ and differentiating both sides with respect to y gives $f_y(x, y) = x + g'(y)$. But $f_y(x, y) = y^2 + x$, so $g'(y) = y^2$ or $g(y) = \frac{1}{3}y^3 + K$. Hence a potential for \mathbf{F} is $f(x, y) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + K$.
5. $\frac{\partial}{\partial y}(1 + 4x^3y^3) = 12x^3y^2 = \frac{\partial}{\partial x}(3x^4y^2)$ and the domain of \mathbf{F} is \mathbb{R}^2 which is open and simply-connected. Thus \mathbf{F} is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = 1 + 4x^3y^3$ implies $f(x, y) = x + x^4y^3 + g(y)$ and $f_y(x, y) = 3x^4y^2 + g'(y)$. But $f_y(x, y) = 3x^4y^2$ implies $g'(y) = 0$. Hence a potential for \mathbf{F} is $f(x, y) = x + x^4y^3 + K$.
6. $\frac{\partial}{\partial y}(y \cos x - \cos y) = \cos x + \sin y$

$$= \frac{\partial}{\partial x}(\sin x + x \sin y)$$

and the domain of \mathbf{F} is \mathbb{R}^2 which is open and simply-connected. Thus \mathbf{F} is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = y \cos x - \cos y$ implies $f(x, y) = y \sin x - x \cos y + g(y)$ and $f_y(x, y) = \sin x + x \sin y + g'(y)$. But $f_y(x, y) = \sin x + x \sin y$, so $g'(y) = 0$. Hence $f(x, y) = y \sin x - x \cos y + K$ is a potential for \mathbf{F} .
7. $\frac{\partial}{\partial y}(e^{2x} + x \sin y) = x \cos y$, $\frac{\partial}{\partial x}(x^2 \cos y) = 2x \cos y$, so \mathbf{F} is not conservative.
8. $\frac{\partial}{\partial y}(ye^{xy} + 4x^3y) = e^{xy}(yx + 1) + 4x^3$

$$= \frac{\partial}{\partial x}(xe^{xy} + x^4)$$

and the domain of \mathbf{F} is \mathbb{R}^2 . Thus \mathbf{F} is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = ye^{xy} + 4x^3y$ implies $f(x, y) = e^{xy} + x^4y + g(y)$ and $f_y(x, y) = xe^{xy} + x^4 + g'(y)$. But $f_y(x, y) = xe^{xy} + x^4$ so $g'(y) = 0$ and $f(x, y) = e^{xy} + x^4y + K$ is a potential for \mathbf{F} .
9. $\frac{\partial}{\partial y}(x + y^2) = 2y = \frac{\partial}{\partial x}(2xy + y^2)$ and the domain of \mathbf{F} is \mathbb{R}^2 . Hence \mathbf{F} is conservative so there exists f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = x + y^2$ implies $f(x, y) = \frac{1}{2}x^2 + xy^2 + g(y)$ and $f_y(x, y) = 2xy + g'(y)$. But $f_y(x, y) = 2xy + y^2$ so $g'(y) = y^2$ or $g(y) = \frac{1}{3}y^3 + K$. Then $f(x, y) = \frac{1}{2}x^2 + xy^2 + \frac{1}{3}y^3 + K$ is a potential for \mathbf{F} .
10. (a) $f_x(x, y) = x$ implies $f(x, y) = \frac{1}{2}x^2 + g(y)$ and $f_y(x, y) = g'(y)$. But $f_y(x, y) = y$ so $g'(y) = \frac{1}{2}y^2 + K$ and $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + K$ (or set $K = 0$).
 (b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 9) - f(-1, 1) = 44$
11. (a) $f_x(x, y) = y$ implies $f(x, y) = xy + g(y)$ and $f_y(x, y) = x + g'(y)$. But $f_y(x, y) = x$ so $f(x, y) = xy$ (setting $K = 0$).
 (b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 8) - f(1, 0) = 16$
12. (a) $f_x(x, y) = 2xy^3$ implies $f(x, y) = x^2y^3 + g(y)$ and $f_y(x, y) = 3x^2y^2 + g'(y)$. But $f_y(x, y) = 3x^2y^2$ so $f(x, y) = x^2y^3$ (setting $K = 0$).
 (b) Since $\mathbf{r}(0) = \langle 0, 1 \rangle$ and $\mathbf{r}(\frac{\pi}{2}) = \langle 1, \frac{1}{4}(\pi^2 + 4) \rangle$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, \frac{1}{4}(\pi^2 + 4)) - f(0, 1) = \frac{1}{64}(\pi^2 + 4)^3.$$
13. (a) $f_x(x, y) = e^{2y}$ implies $f(x, y) = xe^{2y} + g(y)$ and $f_y(x, y) = 2xe^{2y} + g'(y)$. But $f_y(x, y) = 1 + 2xe^{2y}$ so $g'(y) = 1$ and $g(y) = y$ (setting $K = 0$). Thus $f(x, y) = xe^{2y} + y$.
 (b) Since $\mathbf{r}(0) = \langle 0, 1 \rangle$ and $\mathbf{r}(1) = \langle e, 2 \rangle$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(e, 2) - f(0, 1) = (e)e^4 + 2 - 1 = e^5 + 1.$$

14. (a) $f_x(x, y, z) = y$ implies $f(x, y, z) = xy + g(y, z)$ and $f_y(x, y, z) = x + \partial g / \partial y$. But $f_y(x, y, z) = x + z$ so $\partial g / \partial y = z$ and $g(y, z) = yz + h(z)$. Thus $f(x, y, z) = xy + yz + h(z)$ and $f_z(x, y, z) = y + h'(z)$. But $f_z(x, y, z) = y$ so $h'(z) = 0$ or $h(z) = K$. Hence $f(x, y, z) = xy + yz$ (setting $K = 0$).
- (b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(8, 3, -1) - f(2, 1, 4) = 21 - 6 = 15$
15. (a) $f_x(x, y, z) = 2xy^3z^4$ implies $f(x, y, z) = x^2y^3z^4 + g(y, z)$ and $f_y(x, y, z) = 3x^2y^2z^4 + g_y(y, z)$. But $f_y(x, y, z) = 3x^2y^2z^4$, so $g_y(y, z) = h(z)$, and also $f(x, y, z) = x^2y^3z^4 + h(z)$, implying $f_z(x, y, z) = 4x^2y^3z^3 + h'(z)$. But $f_z(x, y, z) = 4x^2y^3z^3$, so $h'(z) = 0$. Hence $f(x, y, z) = x^2y^3z^4$.
- (b) $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ and $\mathbf{r}(2) = \langle 2, 4, 8 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 4, 8) - f(0, 0, 0) = 2^2 \cdot 4^3 \cdot 8^4 = 2^{20}$.
16. (a) $f_x(x, y, z) = 2xz + \sin y$ implies $f(x, y, z) = x^2z + x \sin y + g(y, z)$ and $f_y(x, y, z) = x \cos y + g_y(y, z)$. But $f_y(x, y, z) = x \cos y$ so $g_y(y, z) = 0$ and $f(x, y, z) = x^2z + x \sin y + h(z)$. Thus $f_z(x, y, z) = x^2 + h'(z)$. But $f_z(x, y, z) = x^2$ so $h'(z) = 0$ and $f(x, y, z) = x^2z + x \sin y$ (setting $K = 0$).
- (b) $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$, $\mathbf{r}(2\pi) = \langle 1, 0, 2\pi \rangle$. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 0, 2\pi) - f(1, 0, 0) = 2\pi$.
17. (a) $f_x(x, y, z) = 4xe^z$ implies $f(x, y, z) = 2x^2e^z + g(y, z)$ and $f_y(x, y, z) = g_y(y, z)$. But $f_y(x, y, z) = \cos y$ so $g_y(y, z) = \cos y$ or $g(y, z) = \sin y + h(z)$. Thus $f(x, y, z) = 2x^2e^z + \sin y + h(z)$, and $f_z(x, y, z) = 2x^2e^z + h'(z)$. But $f_z(x, y, z) = 2x^2e^z$ so $h'(z) = 0$ and $f(x, y, z) = 2x^2e^z + \sin y$ (setting $K = 0$).
- (b) $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}(1) = \langle 1, 1, 1 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1, 1) - f(0, 0, 0) = 2e + \sin 1$.
18. Here $\mathbf{F}(x, y) = (2x \sin y) \mathbf{i} + (x^2 \cos y - 3y^2) \mathbf{j}$. Then $f(x, y) = x^2 \sin y - y^3$ is a potential function for \mathbf{F} , that is, $\nabla f = \mathbf{F}$ so \mathbf{F} is conservative and thus its line integral is independent of path. Hence $\int_C 2x \sin y \, dx + (x^2 \cos y - 3y^2) \, dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(5, 1) - f(-1, 0) = 25 \sin 1 - 1$
19. Here $\mathbf{F}(x, y) = (2y^2 - 12x^3y^3) \mathbf{i} + (4xy - 9x^4y^2) \mathbf{j}$. Then $f(x, y) = 2xy^2 - 3x^4y^3$ is a potential function for \mathbf{F} , that is, $\nabla f = \mathbf{F}$. Hence \mathbf{F} is conservative and its line integral is independent of path. $\int_C (2y^2 - 12x^3y^3) \, dx + (4xy - 9x^4y^2) \, dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 2) - f(1, 1) = -1920 - (-1) = -1919$
20. $\mathbf{F}(x, y) = x^2y^3 \mathbf{i} + x^3y^2 \mathbf{j}$, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$. Since $\frac{\partial}{\partial y}(x^2y^3) = 3x^2y^2 = \frac{\partial}{\partial x}(x^3y^2)$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = x^2y^3 \Rightarrow f(x, y) = \frac{1}{3}x^3y^3 + g(y) \Rightarrow f_y = x^3y^2 + g'(y) \Rightarrow g'(y) = 0$, so we can take $f(x, y) = \frac{1}{3}x^3y^3$. Thus $W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 1) - f(0, 0) = \frac{1}{3}(2^3)(1^3) - 0 = \frac{8}{3}$.