13.6 PARAMETRIC SURFACES AND THEIR AREAS

A Click here for answers.

S Click here for solutions.

1–4 ▪ Find a parametric representation for the surface.

1. The part of the hyperboloid $-x^2 - y^2 + z^2 = 1$ that lies below the rectangle $[-1, 1] \times [-3, 3]$

2. The part of the elliptic paraboloid $y = 6 - 3x^2 - 2z^2$ that lies to the right of the xz-plane

3. The part of the cylinder $x^2 + z^2 = 1$ that lies between the planes $y = -1$ and $y = 3$

4. The part of the plane $z = 5$ that lies inside the cylinder $x^2 + y^2 = 16$

5–7 ▪ Find an equation of the tangent plane to the given parametric surface at the specified point. If you have software that graphs parametric surfaces, use a computer to graph the surface and the tangent plane.

5. $x = u^2, \quad y = u - v^2, \quad z = v^2; \quad (1, 0, 1)$

6. $\mathbf{r}(u, v) = uv\,\mathbf{i} + ue^v\,\mathbf{j} + ve^u\,\mathbf{k}; \quad (0, 0, 0)$

7. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + u\cos v\,\mathbf{j} + v\sin u\,\mathbf{k}; \quad (1, 1, 0)$

8. Find the area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices (0, 0), (1, 1), and (0, 1).

9. Set up, but do not evaluate, an integral for the area of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

10. (a) Use the definition of surface area (6) to find the area of the surface with vector equation

$$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + cu\,\mathbf{k}$$

$0 \leqslant u \leqslant h$, $0 \leqslant v \leqslant 2\pi$.

(b) Identify the surface in part (a) by eliminating the parameters and find the surface area using Equation 9.

13.6 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $x = x$, $y = y$, $z = -\sqrt{1 + x^2 + y^2}$, $-1 \leq x \leq 1$, $-3 \leq y \leq 3$

2. $x = x$, $y = 6 - 3x^2 - 2z^2$, $z = z$, $3x^2 + 2z^2 \leq 6$

3. $x = \sin\theta$, $y = y$, $z = \cos\theta$, $0 \leq \theta \leq 2\pi$, $-1 \leq y \leq 3$

4. $x = r\cos\theta$, $y = r\sin\theta$, $z = 5$, $0 \leq r \leq 4$, $0 \leq \theta \leq 2\pi$

5. $x - 2y - 2z + 1 = 0$

6. $x = 0$

7. $(\sin 1)\,x - (\sin 1)\,y - z = 0$

8. $\frac{3}{\sqrt{6}} - \frac{1}{3\sqrt{2}}$

9. $A(S) = 2\int_{-a}^{a}\int_{-(b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} |\mathbf{r}_x \times \mathbf{r}_y|\,dy\,dx$ where

$$|\mathbf{r}_x \times \mathbf{r}_y| = \frac{1}{ab}\sqrt{\frac{a^2b^2\left(a^2b^2 - b^2x^2 - a^2y^2\right) + c^2b^4x^2 + c^2a^4y^2}{a^2b^2 - b^2x^2 - a^2y^2}}$$

10. (a) $\pi h^2\sqrt{c^2+1}$

(b) Disk of radius h centered at the origin

13.6 SOLUTIONS

E Click here for exercises.

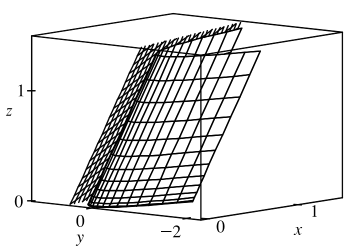
1. Letting x and y be the parameters, parametric equations are $x = x$, $y = y$, $z = -\sqrt{1 + x^2 + y^2}$ (since the surface lies below the rectangle) where $-1 \le x \le 1$ and $-3 \le y \le 3$.
Alternate Solution: Using cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = -\sqrt{1 + r^2}$ where $-1 \le r\cos\theta \le 1$ and $-3 \le r\sin\theta \le 3$.

2. $x = x$, $y = 6 - 3x^2 - 2z^2$, $z = z$ where $3x^2 + 2z^2 \le 6$ since $y \ge 0$. Then the associated vector equation is $\mathbf{r}(x, z) = x\,\mathbf{i} + \left(6 - 3x^2 - 2z^2\right)\mathbf{j} + z\,\mathbf{k}$.

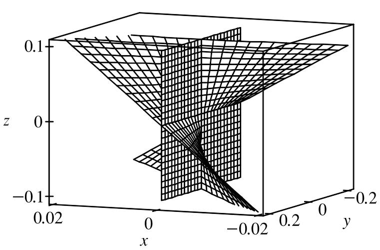
3. In cylindrical coordinates, parametric equations are $x = \sin\theta$, $y = y$, $z = \cos\theta$, $0 \le \theta \le 2\pi$, $-1 \le y \le 3$.

4. The surface is a disk with radius 4 and center $(0, 0, 5)$. Thus, $x = r\cos\theta$, $y = r\sin\theta$, $z = 5$ where $0 \le r \le 4$, $0 \le \theta \le 2\pi$ is a parametric representation of the surface.
Alternate Solution: In rectangular coordinates we could represent the surface as $x = x$, $y = y$, $z = 5$ where $0 \le x^2 + y^2 \le 16$.

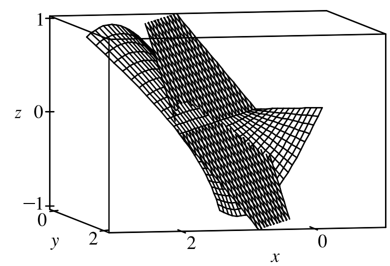
5. $\mathbf{r}(u, v) = \left\langle u^2, u - v^2, v^2 \right\rangle$. $\mathbf{r}_u = \langle 2u, 1, 0\rangle$ and $\mathbf{r}_v = \langle 0, -2v, 2v\rangle$, so $\mathbf{r}_u \times \mathbf{r}_v = \langle 2v, -4uv, -4uv\rangle$. The point $(1, 0, 1)$ corresponds to $u = 1$, $v = \pm 1$. So a normal vector to the surface at $(1, 0, 1)$ is $\pm\langle 2, -4, -4\rangle$ and an equation of the tangent plane is $2x - 4y - 4z = -2$ or $x - 2y - 2z + 1 = 0$.



6. $\mathbf{r}(u, v) = uv\,\mathbf{i} + ue^v\,\mathbf{j} + ve^u\,\mathbf{k}$.
$\mathbf{r}_u = \langle v, e^v, ve^u\rangle$, $\mathbf{r}_v = \langle u, ue^v, e^u\rangle$, and
$\mathbf{r}_u \times \mathbf{r}_v = e^{u+v}(1 - uv)\,\mathbf{i} + e^u(uv - v)\,\mathbf{j} + e^v(uv - u)\,\mathbf{k}$.
The point $(0, 0, 0)$ corresponds to $u = 0$, $v = 0$. Thus a normal vector to the surface at $(0, 0, 0)$ is $\mathbf{i}$, and an equation of the tangent plane is $x = 0$.



7. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + u\cos v\,\mathbf{j} + v\sin u\,\mathbf{k}$.
$\mathbf{r}_u = \langle 1, \cos v, v\cos u\rangle$, $\mathbf{r}_v = \langle 1, -u\sin v, \sin u\rangle$, and
$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v\sin u + uv\cos u\sin v,\; v\cos u - \sin u,\; -u\sin v - \cos v\rangle$$
The point $(1, 1, 0)$ corresponds to $u = 1$, $v = 0$. Thus a normal vector to the surface at $(1, 1, 0)$ is $\langle \sin 1, -\sin 1, -1\rangle$, and an equation of the tangent plane is $(\sin 1)\,x - (\sin 1)\,y - z = 0$.



8. $z = f(x, y) = x + y^2$ with $0 \le x \le y$, $0 \le y \le 1$. Thus, by Formula 9,
$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 1 + 4y^2}\,dA = \int_0^1 \int_0^y \sqrt{2 + 4y^2}\,dx\,dy \\ &= \int_0^1 \left[x\sqrt{2 + 4y^2}\right]_{x=0}^{x=y} dy = \int_0^1 y\sqrt{2 + 4y^2}\,dy \\ &= 2\left(\tfrac{1}{24}\right)\left(2 + 4y^2\right)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(6\sqrt{6} - 2\sqrt{2}\right) \\ &= \tfrac{3}{\sqrt{6}} - \tfrac{1}{3\sqrt{2}} \end{aligned}$$

9. Let S_1 be that portion of S which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$ and S_1 is given by $z = \dfrac{c}{ab}\sqrt{a^2b^2 - b^2x^2 - a^2y^2}$, where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$ or $b^2x^2 + a^2y^2 \le a^2b^2$. Now
$$\begin{aligned} |\mathbf{r}_x \times \mathbf{r}_y| &= \left[1 + \frac{c^2b^2x^2}{a^2}\left(a^2b^2 - b^2x^2 - a^2y^2\right) + \frac{c^2a^2y^2}{b^2}\left(a^2b^2 - b^2x^2 - a^2y^2\right)\right]^{1/2} \\ &= \frac{1}{ab}\sqrt{\frac{a^2b^2\left(a^2b^2 - b^2x^2 - a^2y^2\right) + c^2b^4x^2 + c^2a^4y^2}{a^2b^2 - b^2x^2 - a^2y^2}} \end{aligned}$$
Then $A(S) = 2\int_{-a}^{a}\int_{-(b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} |\mathbf{r}_x \times \mathbf{r}_y|\,dy\,dx$ with $|\mathbf{r}_x \times \mathbf{r}_y|$ given above.
Alternate Solution: Let S_1 be as above. Then in spherical coordinates, $x = a\sin\phi\cos\theta$, $y = b\sin\phi\sin\theta$, and $z = c\cos\phi$, where $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le 2\pi$. Then following Example 9,
$$\mathbf{r}_\phi \times \mathbf{r}_\theta = bc\sin^2\phi\cos\theta\,\mathbf{i} + ac\sin^2\phi\sin\theta\,\mathbf{j} + ab\sin\phi\cos\phi\,\mathbf{k}$$

Hence,

$$A(S) = 2A(S_1) = 2\int_0^{2\pi}\int_0^{\pi/2} \sqrt{\begin{array}{l} b^2c^2\sin^4\phi\cos^2\theta \\ \quad + a^2c^2\sin^4\phi\sin^2\theta \\ \qquad + a^2b^2\sin^2\phi\cos^2\phi \end{array}}\, d\phi\, d\theta$$

10. (a) $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + c\,\mathbf{k}$,

$\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + 0\,\mathbf{k}$, and

$\mathbf{r}_u \times \mathbf{r}_v = -cu\cos v\,\mathbf{i} - cu\sin v\,\mathbf{j} + u\,\mathbf{k} \quad\Rightarrow$

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_0^{2\pi}\int_0^h \sqrt{c^2u^2 + u^2}\, du\, dv = \sqrt{c^2+1}\int_0^{2\pi}\left[\tfrac{1}{2}u^2\right]_0^h dv = \pi h^2\sqrt{c^2+1}$$

(b) $x = u\cos v$, $y = u\sin v$, $z = cu \quad\Rightarrow$ $x^2 + y^2 = (z/c)^2 \quad\Rightarrow\quad z = c\sqrt{x^2+y^2}$, a cone. To find D, notice that $0 \le u \le h \quad\Rightarrow\quad 0 \le z \le ch \quad\Rightarrow$ $0 \le c\sqrt{x^2+y^2} \le ch \quad\Rightarrow\quad 0 \le x^2 + y^2 \le h^2$. So D is a disk of radius h centered at the origin. Therefore

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + \frac{c^2x^2}{x^2+y^2} + \frac{c^2y^2}{x^2+y^2}}\, dA = \iint_D \sqrt{1+c^2}\, dA = \sqrt{1+c^2}\,A(D) = \pi h^2\sqrt{1+c^2}$$