13.8 STOKES' THEOREM

A Click here for answers.

S Click here for solutions.

1–5 ▪ Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

1. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + x\,\mathbf{j} + e^{xy}\cos z\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geqslant 0$, oriented upward

2. $\mathbf{F}(x, y, z) = y^2 z\,\mathbf{i} + xz\,\mathbf{j} + x^2 y^2\,\mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 1$, oriented upward

3. $\mathbf{F}(x, y, z) = yz^3\,\mathbf{i} + \sin(xyz)\,\mathbf{j} + x^3\,\mathbf{k}$,
S is the part of the paraboloid $y = 1 - x^2 - z^2$ that lies to the right of the xz-plane, oriented toward the xz-plane

4. $\mathbf{F}(x, y, z) = (x + \tan^{-1}yz)\,\mathbf{i} + y^2 z\,\mathbf{j} + z\,\mathbf{k}$,
S is the part of the hemisphere $x = \sqrt{9 - y^2 - z^2}$ that lies inside the cylinder $y^2 + z^2 = 4$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + e^z\,\mathbf{j} + xy^2\,\mathbf{k}$,
S consists of the four sides of the pyramid with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, and $(0, 1, 0)$ that lie to the right of the xz-plane, oriented in the direction of the positive y-axis [*Hint:* Use Equation 3.]

6–8 ▪ Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

6. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + 2xy\,\mathbf{j} + 3xy\,\mathbf{k}$,
C is the boundary of the part of the plane $3x + y + z = 3$ in the first octant

7. $\mathbf{F}(x, y, z) = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$,
C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$

8. $\mathbf{F}(x, y, z) = 2z\,\mathbf{i} + 4x\,\mathbf{j} + 5y\,\mathbf{k}$,
C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$

9–12 ▪ Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

9. $\mathbf{F}(x, y, z) = 3y\,\mathbf{i} + 4z\,\mathbf{j} - 6x\,\mathbf{k}$,
S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the xy-plane, oriented upward

10. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$,
S is the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$, oriented upward

11. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the part of the plane $x + y + z = 1$ that lies in the first octant, oriented upward

12. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the helicoid with vector equation
$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

13.8 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. π
2. π
3. $\frac{3\pi}{4}$
4. -4π
5. 0
6. $\frac{7}{2}$
7. $\frac{4}{3}$
8. -4π

13.8 SOLUTIONS

E Click here for exercises.

1. The boundary curve is C: $x^2+y^2=1$, $z=0$ oriented in the counterclockwise direction. The vector equation of C is $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$, $0\le t\le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t))=\cos t\,\mathbf{j}+e^{\cos t\sin t}\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=\cos^2 t$. Hence $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\cos^2 t\,dt=\int_0^{2\pi}\frac{1}{2}(1+\cos 2t)\,dt=\pi$.

2. The paraboloid intersects the cylinder in the circle $x^2+y^2=1$, $z=1$ and the vector equation is $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+\mathbf{k}$, $0\le t\le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t))=\sin^2 t\,\mathbf{i}+\cos t\,\mathbf{j}+\cos^2 t\sin^2 t\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=-\sin^3 t+\cos^2 t$. Hence $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\left(\cos^2 t-\sin^3 t\right)\,dt=\pi$.

3. C is the circle $x^2+z^2=1$, $y=0$ and the vector equation is $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{k}$, $0\le t\le 2\pi$ since the surface is oriented toward the xy-plane. Then $\mathbf{F}(\mathbf{r}(t))=\cos^3 t\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=\cos^4 t$. Hence

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\cos^4 t\,dt=\int_0^{2\pi}\left[\tfrac{3}{8}+\tfrac{1}{2}\cos 2t+\tfrac{1}{8}\cos 4t\right]dt=\tfrac{3\pi}{4}$$

4. C is the circle $y^2+z^2=4$, $x=\sqrt{5}$ with vector equation $\mathbf{r}(t)=\sqrt{5}\,\mathbf{i}+2\cos t\,\mathbf{j}+2\sin t\,\mathbf{k}$, $0\le t\le 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t))=\left[\sqrt{5}+\tan^{-1}(4\cos t\sin t)\right]\mathbf{i}+8\cos^2 t\sin t\,\mathbf{j}+2\sin t\,\mathbf{k}$$

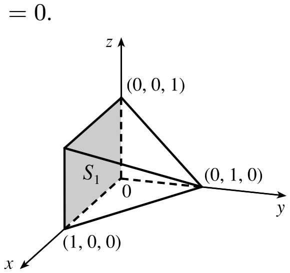
and

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=-16\cos^2 t\sin^2 t+4\sin t\cos t=-2+2\cos 2t+2\sin 2t$$

Thus

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C \mathbf{F}\cdot d\mathbf{r}=2\int_0^{2\pi}(-1+\cos 2t+\sin 2t)\,dt=-4\pi$$

5. Here S consists of the 4 sides of the pyramid but not the base in the xz-plane. Call the base S_1. Then $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the boundary of the base. To avoid calculating four line integrals, apply Stokes' Theorem again. Then $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$. But $\operatorname{curl}\mathbf{F}=(2xy-e^z)\,\mathbf{i}-y^2\,\mathbf{j}-x\,\mathbf{k}$ and $\mathbf{n}=\mathbf{j}$, so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=-y^2=0$ on S_1, $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=0$ and $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=0$.



6. $\operatorname{curl}\mathbf{F}=3x\,\mathbf{i}+(x-3y)\,\mathbf{j}+2y\,\mathbf{k}$, $\mathbf{n}=\dfrac{1}{\sqrt{11}}(3\,\mathbf{i}+\mathbf{j}+\mathbf{k})$

and

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r}&=\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS\\&=\int_0^1\int_0^{3-3x}\tfrac{1}{\sqrt{11}}\left[9x+(x-3y)+2y\right]\left(\sqrt{11}\right)dy\,dx\\&=\int_0^1\int_0^{3-3x}(10x-y)\,dy\,dx\\&=\int_0^1\left[10\left(3x-3x^2\right)-\tfrac{1}{2}(3-3x)^2\right]dx\\&=\left[15x^2-10x^3+\tfrac{3}{2}\left(1-x^3\right)\right]_0^1=\tfrac{7}{2}\end{aligned}$$

7. The triangle is in the plane $2x+2y+z=2$ with normal $\mathbf{n}=\frac{1}{3}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k})$, $\operatorname{curl}\mathbf{F}=x\,\mathbf{i}+(2z-y)\,\mathbf{j}$, $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=\frac{1}{3}(2x+4z-2y)=\frac{1}{3}(8-6x-10y)$ and $dS=3\,dx\,dy$. So

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r}&=\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS\\&=\int_0^1\int_0^{1-x}(8-6x-10y)\,dy\,dx\\&=\int_0^1\left[8(1-x)-6\left(x-x^2\right)-5(1-x)^2\right]dx\\&=\left[8x-7x^2+2x^3+\tfrac{5}{3}(1-x)^3\right]_0^1=\tfrac{4}{3}\end{aligned}$$

8. The curve of intersection is an ellipse in the plane $z=x+4$ with unit normal $\mathbf{n}=\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{k})$ and $\operatorname{curl}\mathbf{F}=5\,\mathbf{i}+2\,\mathbf{j}+4\,\mathbf{k}$ so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=-\frac{1}{\sqrt{2}}$. Then

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r}&=-\iint_S \tfrac{1}{\sqrt{2}}\,dS\\&=-\tfrac{1}{\sqrt{2}}\cdot(\text{surface area of planar ellipse})\\&=-\tfrac{1}{\sqrt{2}}\pi\,(2)\left(2\sqrt{2}\right)=-4\pi\end{aligned}$$

Recall that the area of an ellipse with semiaxes a and b is πab.

9. The boundary curve C is the circle $x^2+y^2=9$, $z=0$ oriented in the counterclockwise direction as viewed from $(0,0,1)$. Then $\mathbf{r}(t)=3\cos t\,\mathbf{i}+3\sin t\,\mathbf{j}$, $0\le t\le 2\pi$, so $\mathbf{F}(\mathbf{r}(t))=9\sin t\,\mathbf{i}-18\cos t\,\mathbf{k}$ and $\mathbf{F}\cdot\mathbf{r}'(t)=-27\sin^2 t$. Thus $\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\left(-27\sin^2 t\right)dt=-27\pi$. Now $\operatorname{curl}\mathbf{F}=-4\,\mathbf{i}+6\,\mathbf{j}-3\,\mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_y=2x\,\mathbf{i}+2y\,\mathbf{j}+\mathbf{k}$, so

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}&=\iint_{x^2+y^2\le 9}(-8x+12y-3)\,dA\\&=\int_0^{2\pi}\int_0^3(-8r\cos\theta+12r\sin\theta-3)\,r\,dr\,d\theta\\&=\int_0^3(-3r)(2\pi)\,dr=-27\pi\end{aligned}$$

10. C: $x^2 + y^2 = a^2$, $z = 0$,

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left(a^2 \sin t \cos t\right) (-a \sin t)\, dt$

$= -\frac{1}{3} a^3 \sin^3 t\Big]_0^{2\pi} = 0$

Then curl $\mathbf{F} = -y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}$,

$$\mathbf{r}_x \times \mathbf{r}_y = \frac{x}{(a^2 - x^2 - y^2)^{1/2}}\,\mathbf{i} + \frac{y}{(a^2 - x^2 - y^2)^{1/2}}\,\mathbf{j} + \mathbf{k}.$$

Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$

$$= \iint_{x^2 + y^2 \le a^2} \left[-\frac{yx}{(a^2 - x^2 - y^2)^{1/2}} - y - x \right] dA$$

$$= -\int_0^a \int_0^{2\pi} \left[\frac{r^2 \cos\theta \sin\theta}{\sqrt{a^2 - r^2}} + r \sin\theta + r\cos\theta \right] r\, d\theta\, dr = 0$$

since $\int_0^{2\pi} \sin\theta\, d\theta = \int_0^{2\pi} \cos\theta\, d\theta = \int_0^{2\pi} \cos\theta \sin\theta\, d\theta = 0$. Notice that for this reason, it's much easier to integrate with respect to θ first.

11. The x-, y-, and z-intercepts of the plane are all 1, so C consists of the three line segments

C_1: $\mathbf{r}_1(t) = (1 - t)\,\mathbf{i} + t\,\mathbf{j}$, $0 \le t \le 1$,

C_2: $\mathbf{r}_2(t) = (1 - t)\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 1$, and

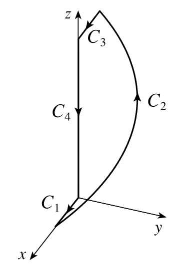
C_3: $\mathbf{r}_3(t) = t\,\mathbf{i} + (1 - t)\,\mathbf{k}$, $0 \le t \le 1$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [t\,\mathbf{i} + (1 - t)\,\mathbf{k}] \cdot (-\mathbf{i} + \mathbf{j})\, dt + \int_0^1 [(1 - t)\,\mathbf{i} + t\,\mathbf{j}] \cdot (-\mathbf{j} + \mathbf{k})\, dt + \int_0^1 [(1 - t)\,\mathbf{j} + t\,\mathbf{k}] \cdot (\mathbf{i} - \mathbf{k})\, dt$$

$= \int_0^1 (-3t)\, dt = -\frac{3}{2}$

Now curl $\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-x} (-3)\, dy\, dx = -\frac{3}{2}$.

12. The equations of the helicoid are



$\mathbf{r}(u, v) = u \cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$. The boundary curve C of the helicoid consists of the four curves C_1: $v = 0$, $0 \le u \le 1$, C_2: $u = 1$, $0 \le v \le \pi$, C_3: $v = \pi$, $u = 1$ to $u = 0$, C_4: $u = 0$, $v = \pi$ to $v = 0$. Then

$\oint_C \mathbf{F} \cdot d\mathbf{r}$

$$= \int_0^1 (u\,\mathbf{k}) \cdot (\mathbf{i})\, du + \int_0^\pi (\sin v\,\mathbf{i} + v\,\mathbf{j} + \cos v\,\mathbf{k}) \cdot (-\sin v\,\mathbf{i} + \cos v\,\mathbf{j} + \mathbf{k})\, dv + \int_1^0 (-u\,\mathbf{k}) \cdot (-\mathbf{i})\, du + \int_\pi^0 (v\,\mathbf{j}) \cdot (\mathbf{k})\, dv$$

$= \int_0^\pi \left(-\sin^2 v + v \cos v + \cos v\right) dv$

$= \left[-\frac{1}{2}(v - \sin v \cos v) + v \sin v + \cos v + \sin v\right]_0^\pi$

$= -\frac{\pi}{2} - 2 = -\frac{1}{2}(\pi + 4)$

Now curl $\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{r}_u \times \mathbf{r}_v = \sin v\,\mathbf{i} - \cos v\,\mathbf{j} + u\,\mathbf{k}$. Hence

$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_0^\pi \int_0^1 (-\sin v + \cos v - u)\, du\, dv$

$= \int_0^\pi \left[-\sin v + \cos v - \frac{1}{2}\right] dv$

$= -2 - \frac{\pi}{2} = -\frac{1}{2}(\pi + 4)$