NONHOMOGENEOUS LINEAR EQUATIONS

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$\boxed{1} \qquad ay'' + by' + cy = G(x)$$

where a, b, and c are constants and G is a continuous function. The related homogeneous equation

$$\boxed{2} \qquad ay'' + by' + cy = 0$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation (1).

3 Theorem The general solution of the nonhomogeneous differential equation (1) can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of Equation 1 and y_c is the general solution of the complementary Equation 2.

Proof All we have to do is verify that if y is any solution of Equation 1, then $y - y_p$ is a solution of the complementary Equation 2. Indeed

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) &= ay'' - ay_p'' + by' - by_p' + cy - cy_p \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= g(x) - g(x) = 0 \end{aligned}$$

■

We know from *Additional Topics: Second-Order Linear Differential Equations* how to solve the complementary equation. (Recall that the solution is $y_c = c_1y_1 + c_2y_2$, where y_1 and y_2 are linearly independent solutions of Equation 2.) Therefore, Theorem 3 says that we know the general solution of the nonhomogeneous equation as soon as we know a particular solution y_p. There are two methods for finding a particular solution: The method of undetermined coefficients is straightforward but works only for a restricted class of functions G. The method of variation of parameters works for every function G but is usually more difficult to apply in practice.

THE METHOD OF UNDETERMINED COEFFICIENTS

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial. It is reasonable to guess that there is a particular solution y_p that is a polynomial of the same degree as G because if y is a polynomial, then $ay'' + by' + cy$ is also a polynomial. We therefore substitute $y_p(x) =$ a polynomial (of the same degree as G) into the differential equation and determine the coefficients.

EXAMPLE 1 Solve the equation $y'' + y' - 2y = x^2$.

SOLUTION The auxiliary equation of $y'' + y' - 2y = 0$ is

$$r^2 + r - 2 = (r - 1)(r + 2) = 0$$

with roots $r = 1, -2$. So the solution of the complementary equation is

$$y_c = c_1 e^x + c_2 e^{-2x}$$

Since $G(x) = x^2$ is a polynomial of degree 2, we seek a particular solution of the form

$$y_p(x) = Ax^2 + Bx + C$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A$ so, substituting into the given differential equation, we have

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

or
$$-2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$$

Polynomials are equal when their coefficients are equal. Thus

$$-2A = 1 \qquad 2A - 2B = 0 \qquad 2A + B - 2C = 0$$

The solution of this system of equations is

$$A = -\tfrac{1}{2} \qquad B = -\tfrac{1}{2} \qquad C = -\tfrac{3}{4}$$

A particular solution is therefore

$$y_p(x) = -\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$

and, by Theorem 3, the general solution is

$$y = y_c + y_p = c_1 e^x + c_2 e^{-2x} - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{3}{4}$$ ■

■■ Figure 1 shows four solutions of the differential equation in Example 1 in terms of the particular solution y_p and the functions $f(x) = e^x$ and $g(x) = e^{-2x}$.

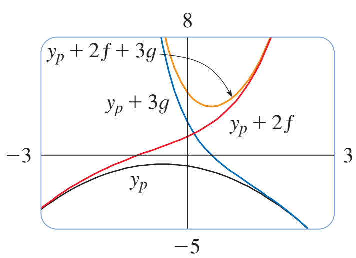


FIGURE 1

If $G(x)$ (the right side of Equation 1) is of the form Ce^{kx}, where C and k are constants, then we take as a trial solution a function of the same form, $y_p(x) = Ae^{kx}$, because the derivatives of e^{kx} are constant multiples of e^{kx}.

EXAMPLE 2 Solve $y'' + 4y = e^{3x}$.

SOLUTION The auxiliary equation is $r^2 + 4 = 0$ with roots $\pm 2i$, so the solution of the complementary equation is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

For a particular solution we try $y_p(x) = Ae^{3x}$. Then $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting into the differential equation, we have

$$9Ae^{3x} + 4(Ae^{3x}) = e^{3x}$$

so $13Ae^{3x} = e^{3x}$ and $A = \frac{1}{13}$. Thus, a particular solution is

$$y_p(x) = \tfrac{1}{13}e^{3x}$$

and the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \tfrac{1}{13}e^{3x}$$ ■

■■ Figure 2 shows solutions of the differential equation in Example 2 in terms of y_p and the functions $f(x) = \cos 2x$ and $g(x) = \sin 2x$. Notice that all solutions approach ∞ as $x \to \infty$ and all solutions resemble sine functions when x is negative.

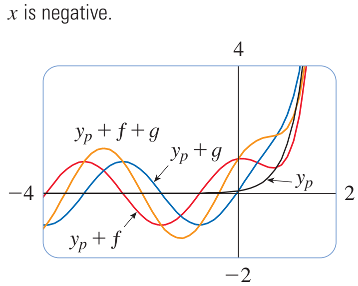


FIGURE 2

If $G(x)$ is either $C \cos kx$ or $C \sin kx$, then, because of the rules for differentiating the sine and cosine functions, we take as a trial particular solution a function of the form

$$y_p(x) = A \cos kx + B \sin kx$$

EXAMPLE 3 Solve $y'' + y' - 2y = \sin x$.

SOLUTION We try a particular solution

$$y_p(x) = A\cos x + B\sin x$$

Then $\qquad y_p' = -A\sin x + B\cos x \qquad y_p'' = -A\cos x - B\sin x$

so substitution in the differential equation gives

$$(-A\cos x - B\sin x) + (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = \sin x$$

or $$(-3A + B)\cos x + (-A - 3B)\sin x = \sin x$$

This is true if

$$-3A + B = 0 \qquad \text{and} \qquad -A - 3B = 1$$

The solution of this system is

$$A = -\tfrac{1}{10} \qquad B = -\tfrac{3}{10}$$

so a particular solution is

$$y_p(x) = -\tfrac{1}{10}\cos x - \tfrac{3}{10}\sin x$$

In Example 1 we determined that the solution of the complementary equation is $y_c = c_1e^x + c_2e^{-2x}$. Thus, the general solution of the given equation is

$$y(x) = c_1e^x + c_2e^{-2x} - \tfrac{1}{10}(\cos x + 3\sin x)$$

If $G(x)$ is a product of functions of the preceding types, then we take the trial solution to be a product of functions of the same type. For instance, in solving the differential equation

$$y'' + 2y' + 4y = x\cos 3x$$

we would try

$$y_p(x) = (Ax + B)\cos 3x + (Cx + D)\sin 3x$$

If $G(x)$ is a sum of functions of these types, we use the easily verified *principle of superposition,* which says that if y_{p_1} and y_{p_2} are solutions of

$$ay'' + by' + cy = G_1(x) \qquad ay'' + by' + cy = G_2(x)$$

respectively, then $y_{p_1} + y_{p_2}$ is a solution of

$$ay'' + by' + cy = G_1(x) + G_2(x)$$

EXAMPLE 4 Solve $y'' - 4y = xe^x + \cos 2x$.

SOLUTION The auxiliary equation is $r^2 - 4 = 0$ with roots ± 2, so the solution of the complementary equation is $y_c(x) = c_1e^{2x} + c_2e^{-2x}$. For the equation $y'' - 4y = xe^x$ we try

$$y_{p_1}(x) = (Ax + B)e^x$$

Then $y_{p_1}' = (Ax + A + B)e^x$, $y_{p_1}'' = (Ax + 2A + B)e^x$, so substitution in the equation gives

$$(Ax + 2A + B)e^x - 4(Ax + B)e^x = xe^x$$

or $$(-3Ax + 2A - 3B)e^x = xe^x$$

Thus, $-3A = 1$ and $2A - 3B = 0$, so $A = -\frac{1}{3}$, $B = -\frac{2}{9}$, and

$$y_{p_1}(x) = \left(-\tfrac{1}{3}x - \tfrac{2}{9}\right)e^x$$

For the equation $y'' - 4y = \cos 2x$, we try

$$y_{p_2}(x) = C\cos 2x + D\sin 2x$$

Substitution gives

$$-4C\cos 2x - 4D\sin 2x - 4(C\cos 2x + D\sin 2x) = \cos 2x$$

or

$$-8C\cos 2x - 8D\sin 2x = \cos 2x$$

Therefore, $-8C = 1$, $-8D = 0$, and

$$y_{p_2}(x) = -\tfrac{1}{8}\cos 2x$$

By the superposition principle, the general solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1e^{2x} + c_2e^{-2x} - \left(\tfrac{1}{3}x + \tfrac{2}{9}\right)e^x - \tfrac{1}{8}\cos 2x \qquad ■$$

■■ In Figure 3 we show the particular solution $y_p = y_{p_1} + y_{p_2}$ of the differential equation in Example 4. The other solutions are given in terms of $f(x) = e^{2x}$ and $g(x) = e^{-2x}$.

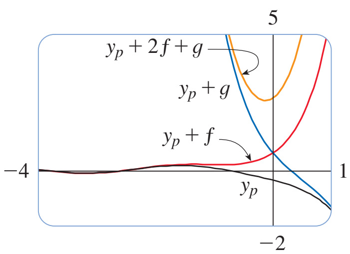


FIGURE 3

Finally we note that the recommended trial solution y_p sometimes turns out to be a solution of the complementary equation and therefore can't be a solution of the nonhomogeneous equation. In such cases we multiply the recommended trial solution by x (or by x^2 if necessary) so that no term in $y_p(x)$ is a solution of the complementary equation.

EXAMPLE 5 Solve $y'' + y = \sin x$.

SOLUTION The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of the complementary equation is

$$y_c(x) = c_1\cos x + c_2\sin x$$

Ordinarily, we would use the trial solution

$$y_p(x) = A\cos x + B\sin x$$

but we observe that it is a solution of the complementary equation, so instead we try

$$y_p(x) = Ax\cos x + Bx\sin x$$

Then

$$y_p'(x) = A\cos x - Ax\sin x + B\sin x + Bx\cos x$$

$$y_p''(x) = -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x$$

■■ The graphs of four solutions of the differential equation in Example 5 are shown in Figure 4.

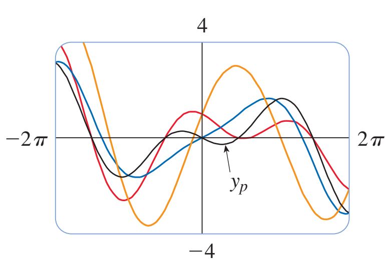


FIGURE 4

Substitution in the differential equation gives

$$y_p'' + y_p = -2A\sin x + 2B\cos x = \sin x$$

so $A = -\frac{1}{2}$, $B = 0$, and

$$y_p(x) = -\tfrac{1}{2}x\cos x$$

The general solution is

$$y(x) = c_1\cos x + c_2\sin x - \tfrac{1}{2}x\cos x \qquad ■$$

We summarize the method of undetermined coefficients as follows:

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation.)

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

EXAMPLE 6 Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x}\cos 3x$.

SOLUTION Here $G(x)$ has the form of part 2 of the summary, where $k = 2$, $m = 3$, and $P(x) = 1$. So, at first glance, the form of the trial solution would be

$$y_p(x) = e^{2x}(A\cos 3x + B\sin 3x)$$

But the auxiliary equation is $r^2 - 4r + 13 = 0$, with roots $r = 2 \pm 3i$, so the solution of the complementary equation is

$$y_c(x) = e^{2x}(c_1\cos 3x + c_2\sin 3x)$$

This means that we have to multiply the suggested trial solution by x. So, instead, we use

$$y_p(x) = xe^{2x}(A\cos 3x + B\sin 3x)$$ ■

THE METHOD OF VARIATION OF PARAMETERS

Suppose we have already solved the homogeneous equation $ay'' + by' + cy = 0$ and written the solution as

$$\boxed{4} \qquad y(x) = c_1y_1(x) + c_2y_2(x)$$

where y_1 and y_2 are linearly independent solutions. Let's replace the constants (or parameters) c_1 and c_2 in Equation 4 by arbitrary functions $u_1(x)$ and $u_2(x)$. We look for a particular solution of the nonhomogeneous equation $ay'' + by' + cy = G(x)$ of the form

$$\boxed{5} \qquad y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

(This method is called **variation of parameters** because we have varied the parameters c_1 and c_2 to make them functions.) Differentiating Equation 5, we get

$$\boxed{6} \qquad y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2')$$

Since u_1 and u_2 are arbitrary functions, we can impose two conditions on them. One condition is that y_p is a solution of the differential equation; we can choose the other condition so as to simplify our calculations. In view of the expression in Equation 6, let's impose the condition that

$$\boxed{7} \qquad u_1'y_1 + u_2'y_2 = 0$$

Then $$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$$

Substituting in the differential equation, we get

$$a(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) = G$$

or

$$\boxed{8} \qquad u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1'y_1' + u_2'y_2') = G$$

But y_1 and y_2 are solutions of the complementary equation, so

$$ay_1'' + by_1' + cy_1 = 0 \qquad \text{and} \qquad ay_2'' + by_2' + cy_2 = 0$$

and Equation 8 simplifies to

$$\boxed{9} \qquad a(u_1'y_1' + u_2'y_2') = G$$

Equations 7 and 9 form a system of two equations in the unknown functions u_1' and u_2'. After solving this system we may be able to integrate to find u_1 and u_2 and then the particular solution is given by Equation 5.

EXAMPLE 7 Solve the equation $y'' + y = \tan x$, $0 < x < \pi/2$.

SOLUTION The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of $y'' + y = 0$ is $c_1 \sin x + c_2 \cos x$. Using variation of parameters, we seek a solution of the form

$$y_p(x) = u_1(x) \sin x + u_2(x) \cos x$$

Then $$y_p' = (u_1' \sin x + u_2' \cos x) + (u_1 \cos x - u_2 \sin x)$$

Set

$$\boxed{10} \qquad u_1' \sin x + u_2' \cos x = 0$$

Then $$y_p'' = u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x$$

For y_p to be a solution we must have

$$\boxed{11} \qquad y_p'' + y_p = u_1' \cos x - u_2' \sin x = \tan x$$

Solving Equations 10 and 11, we get

$$u_1'(\sin^2 x + \cos^2 x) = \cos x \tan x$$

$$u_1' = \sin x \qquad u_1(x) = -\cos x$$

(We seek a particular solution, so we don't need a constant of integration here.) Then, from Equation 10, we obtain

$$u_2' = -\frac{\sin x}{\cos x} u_1' = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

So $$u_2(x) = \sin x - \ln(\sec x + \tan x)$$

■■ Figure 5 shows four solutions of the differential equation in Example 7.

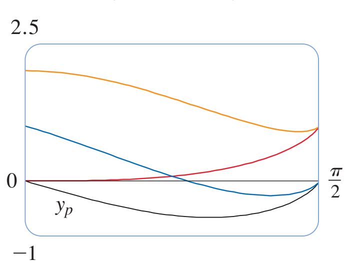


FIGURE 5

(Note that $\sec x + \tan x > 0$ for $0 < x < \pi/2$.) Therefore

$$\begin{aligned} y_p(x) &= -\cos x \sin x + [\sin x - \ln(\sec x + \tan x)] \cos x \\ &= -\cos x \ln(\sec x + \tan x) \end{aligned}$$

and the general solution is

$$y(x) = c_1 \sin x + c_2 \cos x - \cos x \ln(\sec x + \tan x)$$

■

EXERCISES

A Click here for answers. S Click here for solutions.

1–10 ■ Solve the differential equation or initial-value problem using the method of undetermined coefficients.

1. $y'' + 3y' + 2y = x^2$

2. $y'' + 9y = e^{3x}$

3. $y'' - 2y' = \sin 4x$

4. $y'' + 6y' + 9y = 1 + x$

5. $y'' - 4y' + 5y = e^{-x}$

6. $y'' + 2y' + y = xe^{-x}$

7. $y'' + y = e^x + x^3, \quad y(0) = 2, \quad y'(0) = 0$

8. $y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2$

9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$

10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$

11–12 ■ Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

11. $4y'' + 5y' + y = e^x$

12. $2y'' + 3y' + y = 1 + \cos 2x$

13–18 ■ Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

13. $y'' + 9y = e^{2x} + x^2 \sin x$

14. $y'' + 9y' = xe^{-x} \cos \pi x$

15. $y'' + 9y' = 1 + xe^{9x}$

16. $y'' + 3y' - 4y = (x^3 + x)e^x$

17. $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$

18. $y'' + 4y = e^{3x} + x \sin 2x$

19–22 ■ Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

19. $y'' + 4y = x$

20. $y'' - 3y' + 2y = \sin x$

21. $y'' - 2y' + y = e^{2x}$

22. $y'' - y' = e^x$

23–28 ■ Solve the differential equation using the method of variation of parameters.

23. $y'' + y = \sec x, \ 0 < x < \pi/2$

24. $y'' + y = \cot x, \ 0 < x < \pi/2$

25. $y'' - 3y' + 2y = \dfrac{1}{1 + e^{-x}}$

26. $y'' + 3y' + 2y = \sin(e^x)$

27. $y'' - y = \dfrac{1}{x}$

28. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^3}$

ANSWERS

S Click here for solutions.

1. $y = c_1e^{-2x} + c_2e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$

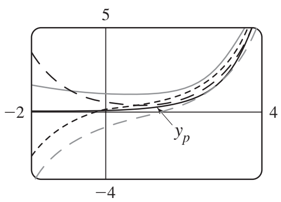
3. $y = c_1 + c_2e^{2x} + \frac{1}{40}\cos 4x - \frac{1}{20}\sin 4x$

5. $y = e^{2x}(c_1\cos x + c_2\sin x) + \frac{1}{10}e^{-x}$

7. $y = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + x^3 - 6x$

9. $y = e^x\left(\frac{1}{2}x^2 - x + 2\right)$

11.



The solutions are all asymptotic to $y_p = e^x/10$ as $x \to \infty$. Except for y_p, all solutions approach either ∞ or $-\infty$ as $x \to -\infty$.

13. $y_p = Ae^{2x} + (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$

15. $y_p = Ax + (Bx + C)e^{9x}$

17. $y_p = xe^{-x}[(Ax^2 + Bx + C)\cos 3x + (Dx^2 + Ex + F)\sin 3x]$

19. $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}x$

21. $y = c_1e^x + c_2xe^x + e^{2x}$

23. $y = (c_1 + x)\sin x + (c_2 + \ln\cos x)\cos x$

25. $y = [c_1 + \ln(1 + e^{-x})]e^x + [c_2 - e^{-x} + \ln(1 + e^{-x})]e^{2x}$

27. $y = \left[c_1 - \frac{1}{2}\int (e^x/x)\,dx\right]e^{-x} + \left[c_2 + \frac{1}{2}\int (e^{-x}/x)\,dx\right]e^x$

SOLUTIONS

1. The auxiliary equation is $r^2 + 3r + 2 = (r+2)(r+1) = 0$, so the complementary solution is $y_c(x) = c_1 e^{-2x} + c_2 e^{-x}$. We try the particular solution $y_p(x) = Ax^2 + Bx + C$, so $y_p' = 2Ax + B$ and $y_p'' = 2A$. Substituting into the differential equation, we have $(2A) + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$ or $2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2$. Comparing coefficients gives $2A = 1$, $6A + 2B = 0$, and $2A + 3B + 2C = 0$, so $A = \frac{1}{2}$, $B = -\frac{3}{2}$, and $C = \frac{7}{4}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$.

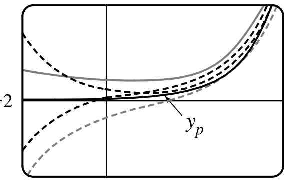
3. The auxiliary equation is $r^2 - 2r = r(r-2) = 0$, so the complementary solution is $y_c(x) = c_1 + c_2 e^{2x}$. Try the particular solution $y_p(x) = A\cos 4x + B\sin 4x$, so $y_p' = -4A\sin 4x + 4B\cos 4x$ and $y_p'' = -16A\cos 4x - 16B\sin 4x$. Substitution into the differential equation gives $(-16A\cos 4x - 16B\sin 4x) - 2(-4A\sin 4x + 4B\cos 4x) = \sin 4x \quad \Rightarrow$ $(-16A - 8B)\cos 4x + (8A - 16B)\sin 4x = \sin 4x$. Then $-16A - 8B = 0$ and $8A - 16B = 1 \quad \Rightarrow \quad A = \frac{1}{40}$ and $B = -\frac{1}{20}$. Thus the general solution is $y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^{2x} + \frac{1}{40}\cos 4x - \frac{1}{20}\sin 4x$.

5. The auxiliary equation is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$, so the complementary solution is $y_c(x) = e^{2x}(c_1\cos x + c_2\sin x)$. Try $y_p(x) = Ae^{-x}$, so $y_p' = -Ae^{-x}$ and $y_p'' = Ae^{-x}$. Substitution gives $Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \quad \Rightarrow \quad 10Ae^{-x} = e^{-x} \quad \Rightarrow \quad A = \frac{1}{10}$. Thus the general solution is $y(x) = e^{2x}(c_1\cos x + c_2\sin x) + \frac{1}{10}e^{-x}$.

7. The auxiliary equation is $r^2 + 1 = 0$ with roots $r = \pm i$, so the complementary solution is $y_c(x) = c_1\cos x + c_2\sin x$. For $y'' + y = e^x$ try $y_{p_1}(x) = Ae^x$. Then $y_{p_1}' = y_{p_1}'' = Ae^x$ and substitution gives $Ae^x + Ae^x = e^x \quad \Rightarrow \quad A = \frac{1}{2}$, so $y_{p_1}(x) = \frac{1}{2}e^x$. For $y'' + y = x^3$ try $y_{p_2}(x) = Ax^3 + Bx^2 + Cx + D$. Then $y_{p_2}' = 3Ax^2 + 2Bx + C$ and $y_{p_2}'' = 6Ax + 2B$. Substituting, we have $6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$, so $A = 1$, $B = 0$, $6A + C = 0 \quad \Rightarrow \quad C = -6$, and $2B + D = 0 \quad \Rightarrow$ $D = 0$. Thus $y_{p_2}(x) = x^3 - 6x$ and the general solution is $y(x) = y_c(x) + y_{p_1}(x) + y_{p_2}(x) = c_1\cos x + c_2\sin x + \frac{1}{2}e^x + x^3 - 6x$. But $2 = y(0) = c_1 + \frac{1}{2} \quad \Rightarrow \quad c_1 = \frac{3}{2}$ and $0 = y'(0) = c_2 + \frac{1}{2} - 6 \quad \Rightarrow \quad c_2 = \frac{11}{2}$. Thus the solution to the initial-value problem is $y(x) = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + x^3 - 6x$.

9. The auxiliary equation is $r^2 - r = 0$ with roots $r = 0$, $r = 1$ so the complementary solution is $y_c(x) = c_1 + c_2 e^x$. Try $y_p(x) = x(Ax + B)e^x$ so that no term in y_p is a solution of the complementary equation. Then $y_p' = (Ax^2 + (2A + B)x + B)e^x$ and $y_p'' = (Ax^2 + (4A + B)x + (2A + 2B))e^x$. Substitution into the differential equation gives $(Ax^2 + (4A + B)x + (2A + 2B))e^x - (Ax^2 + (2A + B)x + B)e^x = xe^x \quad \Rightarrow$ $(2Ax + (2A + B))e^x = xe^x \quad \Rightarrow \quad A = \frac{1}{2}$, $B = -1$. Thus $y_p(x) = \left(\frac{1}{2}x^2 - x\right)e^x$ and the general solution is $y(x) = c_1 + c_2 e^x + \left(\frac{1}{2}x^2 - x\right)e^x$. But $2 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_2 - 1$, so $c_2 = 2$ and $c_1 = 0$. The solution to the initial-value problem is $y(x) = 2e^x + \left(\frac{1}{2}x^2 - x\right)e^x = e^x\left(\frac{1}{2}x^2 - x + 2\right)$.

11. $y_c(x) = c_1 e^{-x/4} + c_2 e^{-x}$. Try $y_p(x) = Ae^x$. Then $10Ae^x = e^x$, so $A = \frac{1}{10}$ and the general solution is $y(x) = c_1 e^{-x/4} + c_2 e^{-x} + \frac{1}{10}e^x$. The solutions are all composed of exponential curves and with the exception of the particular solution (which approaches 0 as $x \to -\infty$), they all approach either ∞ or $-\infty$ as $x \to -\infty$. As $x \to \infty$, all solutions are asymptotic to $y_p = \frac{1}{10}e^x$.



13. Here $y_c(x) = c_1 \cos 3x + c_2 \sin 3x$. For $y'' + 9y = e^{2x}$ try $y_{p_1}(x) = Ae^{2x}$ and for $y'' + 9y = x^2 \sin x$ try $y_{p_2}(x) = (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$. Thus a trial solution is $y_p(x) = y_{p_1}(x) + y_{p_2}(x) = Ae^{2x} + (Bx^2 + Cx + D)\cos x + (Ex^2 + Fx + G)\sin x$.

15. Here $y_c(x) = c_1 + c_2 e^{-9x}$. For $y'' + 9y' = 1$ try $y_{p_1}(x) = Ax$ (since $y = A$ is a solution to the complementary equation) and for $y'' + 9y' = xe^{9x}$ try $y_{p_2}(x) = (Bx + C)e^{9x}$.

17. Since $y_c(x) = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$ we try $y_p(x) = x(Ax^2 + Bx + C)e^{-x} \cos 3x + x(Dx^2 + Ex + F)e^{-x} \sin 3x$ (so that no term of y_p is a solution of the complementary equation).

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u_1' = -\frac{Gy_2}{a\,(y_1 y_2' - y_2 y_1')} \qquad \text{and} \qquad u_2' = \frac{Gy_1}{a\,(y_1 y_2' - y_2 y_1')}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

19. (a) The complementary solution is $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. A particular solution is of the form $y_p(x) = Ax + B$. Thus, $4Ax + 4B = x \;\Rightarrow\; A = \frac{1}{4}$ and $B = 0 \;\Rightarrow\; y_p(x) = \frac{1}{4}x$. Thus, the general solution is $y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x$.

(b) In (a), $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$, so set $y_1 = \cos 2x$, $y_2 = \sin 2x$. Then $y_1 y_2' - y_2 y_1' = 2\cos^2 2x + 2\sin^2 2x = 2$ so $u_1' = -\frac{1}{2}x \sin 2x \;\Rightarrow$ $u_1(x) = -\frac{1}{2}\int x \sin 2x \, dx = -\frac{1}{4}\left(-x\cos 2x + \frac{1}{2}\sin 2x\right)$ [by parts] and $u_2' = \frac{1}{2}x\cos 2x$ $\Rightarrow\; u_2(x) = \frac{1}{2}\int x\cos 2x dx = \frac{1}{4}\left(x \sin 2x + \frac{1}{2}\cos 2x\right)$ [by parts]. Hence $y_p(x) = -\frac{1}{4}\left(-x\cos 2x + \frac{1}{2}\sin 2x\right)\cos 2x + \frac{1}{4}\left(x\sin 2x + \frac{1}{2}\cos 2x\right)\sin 2x = \frac{1}{4}x$. Thus $y(x) = y_c(x) + y_p(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x$.

21. (a) $r^2 - r = r(r-1) = 0 \;\Rightarrow\; r = 0, 1$, so the complementary solution is $y_c(x) = c_1 e^x + c_2 xe^x$. A particular solution is of the form $y_p(x) = Ae^{2x}$. Thus $4Ae^{2x} - 4Ae^{2x} + Ae^{2x} = e^{2x} \;\Rightarrow\; Ae^{2x} = e^{2x} \;\Rightarrow\; A = 1$ $\Rightarrow\; y_p(x) = e^{2x}$. So a general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 xe^x + e^{2x}$.

(b) From (a), $y_c(x) = c_1 e^x + c_2 xe^x$, so set $y_1 = e^x$, $y_2 = xe^x$. Then, $y_1 y_2' - y_2 y_1' = e^{2x}(1+x) - xe^{2x} = e^{2x}$ and so $u_1' = -xe^x \;\Rightarrow\; u_1(x) = -\int xe^x \, dx = -(x-1)e^x$ [by parts] and $u_2' = e^x \;\Rightarrow$ $u_2(x) = \int e^x \, dx = e^x$. Hence $y_p(x) = (1-x)e^{2x} + xe^{2x} = e^{2x}$ and the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 xe^x + e^{2x}$.

23. As in Example 6, $y_c(x) = c_1 \sin x + c_2 \cos x$, so set $y_1 = \sin x$, $y_2 = \cos x$. Then $y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1$, so $u_1' = -\dfrac{\sec x \cos x}{-1} = 1 \quad \Rightarrow \quad u_1(x) = x$ and $u_2' = \dfrac{\sec x \sin x}{-1} = -\tan x \quad \Rightarrow \quad u_2(x) = -\int \tan x dx = \ln|\cos x| = \ln(\cos x)$ on $0 < x < \frac{\pi}{2}$. Hence $y_p(x) = x \sin x + \cos x \ln(\cos x)$ and the general solution is $y(x) = (c_1 + x)\sin x + [c_2 + \ln(\cos x)]\cos x$.

25. $y_1 = e^x$, $y_2 = e^{2x}$ and $y_1 y_2' - y_2 y_1' = e^{3x}$. So $u_1' = \dfrac{-e^{2x}}{(1+e^{-x})e^{3x}} = -\dfrac{e^{-x}}{1+e^{-x}}$ and $u_1(x) = \displaystyle\int -\frac{e^{-x}}{1+e^{-x}}\,dx = \ln(1+e^{-x})$. $u_2' = \dfrac{e^x}{(1+e^{-x})e^{3x}} = \dfrac{e^x}{e^{3x}+e^{2x}}$ so $u_2(x) = \displaystyle\int \frac{e^x}{e^{3x}+e^{2x}}\,dx = \ln\left(\frac{e^x+1}{e^x}\right) - e^{-x} = \ln(1+e^{-x}) - e^{-x}$. Hence $y_p(x) = e^x \ln(1+e^{-x}) + e^{2x}[\ln(1+e^{-x}) - e^{-x}]$ and the general solution is $y(x) = [c_1 + \ln(1+e^{-x})]e^x + [c_2 - e^{-x} + \ln(1+e^{-x})]e^{2x}$.

27. $y_1 = e^{-x}$, $y_2 = e^x$ and $y_1 y_2' - y_2 y_1' = 2$. So $u_1' = -\dfrac{e^x}{2x}$, $u_2' = \dfrac{e^{-x}}{2x}$ and $y_p(x) = -e^{-x}\displaystyle\int \frac{e^x}{2x}\,dx + e^x \int \frac{e^{-x}}{2x}\,dx$. Hence the general solution is $y(x) = \left(c_1 - \displaystyle\int \frac{e^x}{2x}\,dx\right)e^{-x} + \left(c_2 + \int \frac{e^{-x}}{2x}\,dx\right)e^x$.