USING SERIES TO SOLVE DIFFERENTIAL EQUATIONS

Many differential equations can't be solved explicitly in terms of finite combinations of simple familiar functions. This is true even for a simple-looking equation like

$$y'' - 2xy' + y = 0 \tag{1}$$

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quantum mechanics. In such a case we use the method of power series; that is, we look for a solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients $c_0, c_1, c_2, \ldots$.

Before using power series to solve Equation 1, we illustrate the method on the simpler equation $y'' + y = 0$ in Example 1.

EXAMPLE 1 Use power series to solve the equation $y'' + y = 0$.

SOLUTION We assume there is a solution of the form

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n \tag{2}$$

We can differentiate power series term by term, so

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = 2c_2 + 2 \cdot 3c_3 x + \cdots = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \tag{3}$$

In order to compare the expressions for y and y'' more easily, we rewrite y'' as follows:

■■ By writing out the first few terms of (4), you can see that it is the same as (3). To obtain (4) we replaced n by $n + 2$ and began the summation at 0 instead of 2.

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n \tag{4}$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + c_n] x^n = 0 \tag{5}$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore, the coefficients of x^n in Equation 5 must be 0:

$$(n+2)(n+1) c_{n+2} + c_n = 0$$

$$\boxed{6} \qquad c_{n+2} = -\frac{c_n}{(n+1)(n+2)} \qquad n = 0, 1, 2, 3, \ldots$$

Equation 6 is called a *recursion relation*. If c_0 and c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting $n = 0, 1, 2, 3, \ldots$ in succession.

$$\text{Put } n = 0: \qquad c_2 = -\frac{c_0}{1 \cdot 2}$$

$$\text{Put } n = 1: \qquad c_3 = -\frac{c_1}{2 \cdot 3}$$

$$\text{Put } n = 2: \qquad c_4 = -\frac{c_2}{3 \cdot 4} = \frac{c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{c_0}{4!}$$

$$\text{Put } n = 3: \qquad c_5 = -\frac{c_3}{4 \cdot 5} = \frac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{c_1}{5!}$$

$$\text{Put } n = 4: \qquad c_6 = -\frac{c_4}{5 \cdot 6} = -\frac{c_0}{4!\, 5 \cdot 6} = -\frac{c_0}{6!}$$

$$\text{Put } n = 5: \qquad c_7 = -\frac{c_5}{6 \cdot 7} = -\frac{c_1}{5!\, 6 \cdot 7} = -\frac{c_1}{7!}$$

By now we see the pattern:

$$\text{For the even coefficients, } \; c_{2n} = (-1)^n \frac{c_0}{(2n)!}$$

$$\text{For the odd coefficients, } \; c_{2n+1} = (-1)^n \frac{c_1}{(2n+1)!}$$

Putting these values back into Equation 2, we write the solution as

$$\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\
&= c_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \right) \\
&\qquad + c_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \right) \\
&= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Notice that there are two arbitrary constants, c_0 and c_1. ■

NOTE 1 ▫ We recognize the series obtained in Example 1 as being the Maclaurin series for $\cos x$ and $\sin x$. (See Equations 8.7.17 and 8.7.16.) Therefore, we could write the solution as

$$y(x) = c_0 \cos x + c_1 \sin x$$

But we are not usually able to express power series solutions of differential equations in terms of known functions.

EXAMPLE 2 Solve $y'' - 2xy' + y = 0$.

SOLUTION We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then
$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

and
$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

as in Example 1. Substituting in the differential equation, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2n c_n x^n = \sum_{n=0}^{\infty} 2n c_n x^n$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (2n-1)c_n]x^n = 0$$

This equation is true if the coefficient of x^n is 0:

$$(n+2)(n+1)c_{n+2} - (2n-1)c_n = 0$$

$$\boxed{7} \qquad c_{n+2} = \frac{2n-1}{(n+1)(n+2)} c_n \qquad n = 0, 1, 2, 3, \ldots$$

We solve this recursion relation by putting $n = 0, 1, 2, 3, \ldots$ successively in Equation 7:

Put $n = 0$: $\quad c_2 = \dfrac{-1}{1 \cdot 2} c_0$

Put $n = 1$: $\quad c_3 = \dfrac{1}{2 \cdot 3} c_1$

Put $n = 2$: $\quad c_4 = \dfrac{3}{3 \cdot 4} c_2 = -\dfrac{3}{1 \cdot 2 \cdot 3 \cdot 4} c_0 = -\dfrac{3}{4!} c_0$

Put $n = 3$: $\quad c_5 = \dfrac{5}{4 \cdot 5} c_3 = \dfrac{1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} c_1 = \dfrac{1 \cdot 5}{5!} c_1$

Put $n = 4$: $\quad c_6 = \dfrac{7}{5 \cdot 6} c_4 = -\dfrac{3 \cdot 7}{4!\, 5 \cdot 6} c_0 = -\dfrac{3 \cdot 7}{6!} c_0$

Put $n = 5$: $\quad c_7 = \dfrac{9}{6 \cdot 7} c_5 = \dfrac{1 \cdot 5 \cdot 9}{5!\, 6 \cdot 7} c_1 = \dfrac{1 \cdot 5 \cdot 9}{7!} c_1$

Put $n = 6$: $\quad c_8 = \dfrac{11}{7 \cdot 8} c_6 = -\dfrac{3 \cdot 7 \cdot 11}{8!} c_0$

Put $n = 7$: $\quad c_9 = \dfrac{13}{8 \cdot 9} c_7 = \dfrac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1$

In general, the even coefficients are given by

$$c_{2n} = -\frac{3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n-5)}{(2n)!} c_0$$

and the odd coefficients are given by

$$c_{2n+1} = \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} c_1$$

The solution is

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \\ &= c_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{3 \cdot 7}{6!}x^6 - \frac{3 \cdot 7 \cdot 11}{8!}x^8 - \cdots\right) \\ &\quad + c_1\left(x + \frac{1}{3!}x^3 + \frac{1 \cdot 5}{5!}x^5 + \frac{1 \cdot 5 \cdot 9}{7!}x^7 + \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!}x^9 + \cdots\right) \end{aligned}$$

or

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$$\begin{aligned} y = c_0\left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{(2n)!} x^{2n}\right) \\ + c_1\left(x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}\right) \end{aligned}$$

■

NOTE 2 ▫ In Example 2 we had to assume that the differential equation had a series solution. But now we could verify directly that the function given by Equation 8 is indeed a solution.

NOTE 3 ▫ Unlike the situation of Example 1, the power series that arise in the solution of Example 2 do not define elementary functions. The functions

$$y_1(x) = 1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{(2n)!} x^{2n}$$

and
$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

are perfectly good functions but they can't be expressed in terms of familiar functions. We can use these power series expressions for y_1 and y_2 to compute approximate values of the functions and even to graph them. Figure 1 shows the first few partial sums $T_0, T_2, T_4, \ldots$ (Taylor polynomials) for $y_1(x)$, and we see how they converge to y_1. In this way we can graph both y_1 and y_2 in Figure 2.

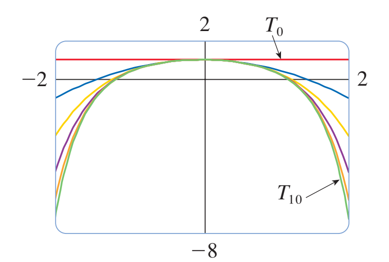

FIGURE 1

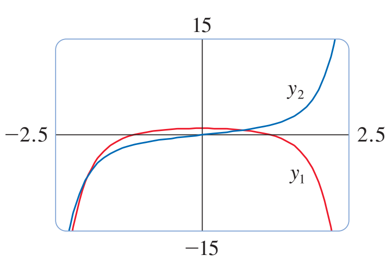

FIGURE 2

NOTE 4 ▫ If we were asked to solve the initial-value problem

$$y'' - 2xy' + y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

we would observe that

$$c_0 = y(0) = 0 \qquad c_1 = y'(0) = 1$$

This would simplify the calculations in Example 2, since all of the even coefficients would be 0. The solution to the initial-value problem is

$$y(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

EXERCISES

A Click here for answers. S Click here for solutions.

1–11 ■ Use power series to solve the differential equation.

1. $y' - y = 0$

2. $y' = xy$

3. $y' = x^2 y$

4. $(x - 3)y' + 2y = 0$

5. $y'' + xy' + y = 0$

6. $y'' = y$

7. $(x^2 + 1)y'' + xy' - y = 0$

8. $y'' = xy$

9. $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$

10. $y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0$

11. $y'' + x^2 y' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1$

12. The solution of the initial-value problem

$$x^2 y'' + xy' + x^2 y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

is called a Bessel function of order 0.

(a) Solve the initial-value problem to find a power series expansion for the Bessel function.

(b) Graph several Taylor polynomials until you reach one that looks like a good approximation to the Bessel function on the interval $[-5, 5]$.

ANSWERS

S Click here for solutions.

1. $c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x$ **3.** $c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 e^{x^3/3}$

5. $c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$

7. $c_0 + c_1 x + c_0 \frac{x^2}{2} + c_0 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!}{2^{2n-2} n!(n-2)!} x^{2n}$

9. $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2}$

11. $x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!} x^{3n+1}$

SOLUTIONS

1. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and the given equation, $y' - y = 0$, becomes

$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$. Replacing n by $n+1$ in the first sum gives $\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$, so

$\sum_{n=0}^{\infty} [(n+1)c_{n+1} - c_n] x^n = 0$. Equating coefficients gives $(n+1)c_{n+1} - c_n = 0$, so the recursion relation is

$c_{n+1} = \dfrac{c_n}{n+1}$, $n = 0, 1, 2, \ldots$. Then $c_1 = c_0$, $c_2 = \frac{1}{2} c_1 = \frac{c_0}{2}$, $c_3 = \frac{1}{3} c_2 = \frac{1}{3} \cdot \frac{1}{2} c_0 = \frac{c_0}{3!}$, $c_4 = \frac{1}{4} c_3 = \frac{c_0}{4!}$, and

in general, $c_n = \dfrac{c_0}{n!}$. Thus, the solution is $y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!} x^n = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x$.

3. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ and

$-x^2 y = -\sum_{n=0}^{\infty} c_n x^{n+2} = -\sum_{n=2}^{\infty} c_{n-2} x^n$. Hence, the equation $y' = x^2 y$ becomes

$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0$ or $c_1 + 2c_2 x + \sum_{n=2}^{\infty} [(n+1)c_{n+1} - c_{n-2}] x^n = 0$. Equating coefficients

gives $c_1 = c_2 = 0$ and $c_{n+1} = \dfrac{c_{n-2}}{n+1}$ for $n = 2, 3, \ldots$. But $c_1 = 0$, so $c_4 = 0$ and

$c_7 = 0$ and in general $c_{3n+1} = 0$. Similarly $c_2 = 0$ so $c_{3n+2} = 0$. Finally $c_3 = \dfrac{c_0}{3}$,

$c_6 = \dfrac{c_3}{6} = \dfrac{c_0}{6 \cdot 3} = \dfrac{c_0}{3^2 \cdot 2!}$, $c_9 = \dfrac{c_6}{9} = \dfrac{c_0}{9 \cdot 6 \cdot 3} = \dfrac{c_0}{3^3 \cdot 3!}$, $\ldots$, and $c_{3n} = \dfrac{c_0}{3^n \cdot n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} = \sum_{n=0}^{\infty} \frac{c_0}{3^n \cdot n!} x^{3n} = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}.$$

5. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential

equation becomes $\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$ or

$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + n c_n + c_n] x^n \left(\text{since } \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n \right)$. Equating coefficients gives

$(n+2)(n+1)c_{n+2} + (n+1)c_n = 0$, thus the recursion relation is $c_{n+2} = \dfrac{-(n+1)c_n}{(n+2)(n+1)} = -\dfrac{c_n}{n+2}$,

$n = 0, 1, 2, \ldots$. Then the even coefficients are given by $c_2 = -\dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = \dfrac{c_0}{2 \cdot 4}$, $c_6 = -\dfrac{c_4}{6} = -\dfrac{c_0}{2 \cdot 4 \cdot 6}$,

and in general, $c_{2n} = (-1)^n \dfrac{c_0}{2 \cdot 4 \cdot \cdots \cdot 2n} = \dfrac{(-1)^n c_0}{2^n n!}$. The odd coefficients are $c_3 = -\dfrac{c_1}{3}$, $c_5 = -\dfrac{c_3}{5} = \dfrac{c_1}{3 \cdot 5}$,

$c_7 = -\dfrac{c_5}{7} = -\dfrac{c_1}{3 \cdot 5 \cdot 7}$, and in general, $c_{2n+1} = (-1)^n \dfrac{c_1}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} = \dfrac{(-2)^n n! \, c_1}{(2n+1)!}$. The solution is

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}.$$

7. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$, $xy' = \sum_{n=0}^{\infty} n c_n x^n$ and

$(x^2 + 1) y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential equation becomes

$\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + [n(n-1) + n - 1] c_n] x^n = 0$. The recursion relation is $c_{n+2} = -\dfrac{(n-1) c_n}{n+2}$,

$n = 0, 1, 2, \ldots$. Given c_0 and c_1, $c_2 = \dfrac{c_0}{2}$, $c_4 = -\dfrac{c_2}{4} = -\dfrac{c_0}{2^2 \cdot 2!}$, $c_6 = -\dfrac{3c_4}{6} = (-1)^2 \dfrac{3c_0}{2^3 \cdot 3!}, \ldots,$

$c_{2n} = (-1)^{n-1} \dfrac{1 \cdot 3 \cdot \cdots \cdot (2n-3)\ c_0}{2^n\, n!} = (-1)^{n-1} \dfrac{(2n-3)!\, c_0}{2^n 2^{n-2}\, n!\, (n-2)!} = (-1)^{n-1} \dfrac{(2n-3)!\, c_0}{2^{2n-2}\, n!\, (n-2)!}$ for

$n = 2, 3, \ldots$. $c_3 = \dfrac{0 \cdot c_1}{3} = 0 \quad \Rightarrow \quad c_{2n+1} = 0$ for $n = 1, 2, \ldots$. Thus the solution is

$y(x) = c_0 + c_1 x + c_0 \dfrac{x^2}{2} + c_0 \sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n-1}(2n-3)!}{2^{2n-2}\, n!\, (n-2)!}\, x^{2n}.$

9. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x \sum\limits_{n=1}^{\infty} n c_n x^{n-1} = -\sum\limits_{n=1}^{\infty} n c_n x^n = -\sum\limits_{n=0}^{\infty} n c_n x^n$,

$y''(x) = \sum\limits_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$, and the equation $y'' - xy' - y = 0$ becomes

$\sum\limits_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n - c_n] x^n = 0$. Thus, the recursion relation is

$c_{n+2} = \dfrac{nc_n + c_n}{(n+2)(n+1)} = \dfrac{c_n(n+1)}{(n+2)(n+1)} = \dfrac{c_n}{n+2}$ for $n = 0, 1, 2, \ldots$. One of the given conditions is

$y(0) = 1$. But $y(0) = \sum\limits_{n=0}^{\infty} c_n (0)^n = c_0 + 0 + 0 + \cdots = c_0$, so $c_0 = 1$. Hence, $c_2 = \dfrac{c_0}{2} = \dfrac{1}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{1}{2 \cdot 4}$,

$c_6 = \dfrac{c_4}{6} = \dfrac{1}{2 \cdot 4 \cdot 6}, \ldots, c_{2n} = \dfrac{1}{2^n n!}$. The other given condition is $y'(0) = 0$. But

$y'(0) = \sum\limits_{n=1}^{\infty} n c_n (0)^{n-1} = c_1 + 0 + 0 + \cdots = c_1$, so $c_1 = 0$. By the recursion relation, $c_3 = \dfrac{c_1}{3} = 0$, $c_5 = 0, \ldots,$

$c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Thus, the solution to the initial-value problem is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = e^{x^2/2}$$

11. Assuming that $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$, we have $xy = x \sum\limits_{n=0}^{\infty} c_n x^n = \sum\limits_{n=0}^{\infty} c_n x^{n+1}$,

$x^2 y' = x^2 \sum\limits_{n=1}^{\infty} n c_n x^{n-1} = \sum\limits_{n=0}^{\infty} n c_n x^{n+1}$,

$$\begin{aligned} y''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} \quad \text{[replace } n \text{ with } n+3\text{]} \\ &= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1}, \end{aligned}$$

and the equation $y'' + x^2 y' + xy = 0$ becomes $2c_2 + \sum\limits_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} + nc_n + c_n]\, x^{n+1} = 0$.

So $c_2 = 0$ and the recursion relation is $c_{n+3} = \dfrac{-nc_n - c_n}{(n+3)(n+2)} = -\dfrac{(n+1)c_n}{(n+3)(n+2)}$, $n = 0, 1, 2, \ldots$.

But $c_0 = y(0) = 0 = c_2$ and by the recursion relation, $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \ldots$.

Also, $c_1 = y'(0) = 1$, so

$c_4 = -\dfrac{2c_1}{4 \cdot 3} = -\dfrac{2}{4 \cdot 3}$, $c_7 = -\dfrac{5c_4}{7 \cdot 6} = (-1)^2 \dfrac{2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = (-1)^2 \dfrac{2^2 5^2}{7!}, \ldots,$

$c_{3n+1} = (-1)^n \dfrac{2^2 5^2 \cdot \cdots \cdot (3n-1)^2}{(3n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty} \left[(-1)^n \frac{2^2 5^2 \cdot \cdots \cdot (3n-1)^2 x^{3n+1}}{(3n+1)!} \right]$$