

# CHALLENGE PROBLEMS

## CHAPTER 3

**A** [Click here for answers.](#)

**S** [Click here for solutions.](#)

1. Show that

$$\frac{d^n}{dx^n} (e^{ax} \sin bx) = r^n e^{ax} \sin(bx + n\theta)$$

where  $a$  and  $b$  are positive numbers,  $r^2 = a^2 + b^2$ , and  $\theta = \tan^{-1}(b/a)$ .

2. Show that  $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$ .

3. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \arctan \frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

show that  $y' = \frac{1}{a + \cos x}$ .

4. For which positive numbers  $a$  is it true that  $a^x \geq 1 + x$  for all  $x$ ?
5. For what value of  $k$  does the equation  $e^{2x} = k\sqrt{x}$  have exactly one solution?
6. A peach pie is taken out of the oven at 5:00 P.M. At that time it is piping hot: 100°C. At 5:10 P.M. its temperature is 80°C; at 5:20 P.M. it is 65°C. What is the temperature of the room?



**ANSWERS**

---

**S Solutions**

5.  $2\sqrt{e}$

**SOLUTIONS**
**E Exercises**

1. Consider the statement that  $\frac{d^n}{dx^n}(e^{ax} \sin bx) = r^n e^{ax} \sin(bx + n\theta)$ . For  $n = 1$ ,

$$\frac{d}{dx}(e^{ax} \sin bx) = ae^{ax} \sin bx + be^{ax} \cos bx, \text{ and}$$

$$\begin{aligned} re^{ax} \sin(bx + \theta) &= re^{ax} [\sin bx \cos \theta + \cos bx \sin \theta] = re^{ax} \left( \frac{a}{r} \sin bx + \frac{b}{r} \cos bx \right) \\ &= ae^{ax} \sin bx + be^{ax} \cos bx \end{aligned}$$

$$\text{since } \tan \theta = \frac{b}{a} \Rightarrow \sin \theta = \frac{b}{r} \text{ and } \cos \theta = \frac{a}{r}.$$

So the statement is true for  $n = 1$ . Assume it is true for  $n = k$ . Then

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(e^{ax} \sin bx) &= \frac{d}{dx} [r^k e^{ax} \sin(bx + k\theta)] = r^k ae^{ax} \sin(bx + k\theta) + r^k e^{ax} b \cos(bx + k\theta) \\ &= r^k e^{ax} [a \sin(bx + k\theta) + b \cos(bx + k\theta)] \end{aligned}$$

But

$$\begin{aligned} \sin[bx + (k+1)\theta] &= \sin[(bx + k\theta) + \theta] = \sin(bx + k\theta) \cos \theta + \sin \theta \cos(bx + k\theta) \\ &= \frac{a}{r} \sin(bx + k\theta) + \frac{b}{r} \cos(bx + k\theta) \end{aligned}$$

Hence,  $a \sin(bx + k\theta) + b \cos(bx + k\theta) = r \sin[bx + (k+1)\theta]$ . So

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(e^{ax} \sin bx) &= r^k e^{ax} [a \sin(bx + k\theta) + b \cos(bx + k\theta)] = r^k e^{ax} [r \sin(bx + (k+1)\theta)] \\ &= r^{k+1} e^{ax} [\sin(bx + (k+1)\theta)] \end{aligned}$$

Therefore, the statement is true for all  $n$  by mathematical induction.

3.  $y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \arctan \frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$ . Let  $k = a + \sqrt{a^2 - 1}$ . Then

$$\begin{aligned} y' &= \frac{1}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \cdot \frac{1}{1 + \sin^2 x / (k + \cos x)^2} \cdot \frac{\cos x(k + \cos x) + \sin^2 x}{(k + \cos x)^2} \\ &= \frac{1}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \cdot \frac{k \cos x + \cos^2 x + \sin^2 x}{(k + \cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \cdot \frac{k \cos x + 1}{k^2 + 2k \cos x + 1} \\ &= \frac{k^2 + 2k \cos x + 1 - 2k \cos x - 2}{\sqrt{a^2 - 1}(k^2 + 2k \cos x + 1)} = \frac{k^2 - 1}{\sqrt{a^2 - 1}(k^2 + 2k \cos x + 1)} \end{aligned}$$

But  $k^2 = 2a^2 + 2a\sqrt{a^2 - 1} - 1 = 2a(a + \sqrt{a^2 - 1}) - 1 = 2ak - 1$ , so  $k^2 + 1 = 2ak$ , and

$$k^2 - 1 = 2(ak - 1). \text{ So } y' = \frac{2(ak - 1)}{\sqrt{a^2 - 1}(2ak + 2k \cos x)} = \frac{ak - 1}{\sqrt{a^2 - 1}k(a + \cos x)}. \text{ But}$$

$$ak - 1 = a^2 + a\sqrt{a^2 - 1} - 1 = k\sqrt{a^2 - 1}, \text{ so } y' = 1/(a + \cos x).$$

5. Let  $f(x) = e^{2x}$  and  $g(x) = k\sqrt{x}$  ( $k > 0$ ). From the graphs of  $f$  and  $g$ , we see that  $f$  will intersect  $g$  exactly once when  $f$  and  $g$  share a tangent line. Thus, we must have  $f = g$  and  $f' = g'$  at  $x = a$ .

$$f(a) = g(a) \Rightarrow e^{2a} = k\sqrt{a} \quad \mathbf{(1)} \quad \text{and} \quad f'(a) = g'(a) \Rightarrow$$

$$2e^{2a} = \frac{k}{2\sqrt{a}} \Rightarrow e^{2a} = \frac{k}{4\sqrt{a}}. \quad \text{So we must have } k\sqrt{a} = \frac{k}{4\sqrt{a}}$$

$$\Rightarrow (\sqrt{a})^2 = \frac{k}{4k} \Rightarrow a = \frac{1}{4}. \quad \text{From } \mathbf{(1)}, e^{2(1/4)} = k\sqrt{1/4} \Rightarrow k = 2e^{1/2} = 2\sqrt{e} \approx 3.297.$$

