

8.3 THE INTEGRAL AND COMPARISON TESTS

EXAMPLE A Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ for convergence or divergence.

SOLUTION The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2 + 1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \tan^{-1} x \Big|_1^t \\ &= \lim_{t \rightarrow \infty} \left(\tan^{-1} t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Thus, $\int_1^{\infty} 1/(x^2 + 1) dx$ is a convergent integral and so, by the Integral Test, the series $\sum 1/(n^2 + 1)$ is convergent. ■

EXAMPLE B Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \quad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n \rightarrow \infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}} \\ &= \lim_{n \rightarrow \infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{2 + 0}{2\sqrt{0 + 1}} = 1 \end{aligned}$$

Since $\sum b_n = 2 \sum 1/n^{1/2}$ is divergent (p -series with $p = \frac{1}{2} < 1$), the given series diverges by the Limit Comparison Test. ■