AREA OF A SURFACE OF REVOLUTION

A surface of revolution is formed when a curve is rotated about a line. Such a surface is the lateral boundary of a solid of revolution of the type discussed in Sections 7.2 and 7.3.

We want to define the area of a surface of revolution in such a way that it corresponds to our intuition. If the surface area is A, we can imagine that painting the surface would require the same amount of paint as does a flat region with area A.

Let's start with some simple surfaces. The lateral surface area of a circular cylinder with radius r and height h is taken to be $A = 2\pi rh$ because we can imagine cutting the cylinder and unrolling it (as in Figure 1) to obtain a rectangle with dimensions $2\pi r$ and h.

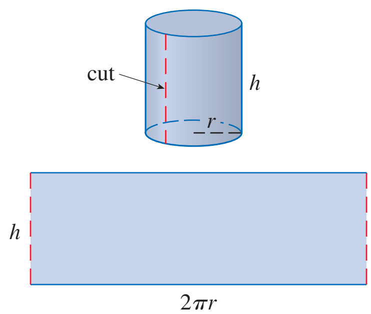


FIGURE 1

Likewise, we can take a circular cone with base radius r and slant height l, cut it along the dashed line in Figure 2, and flatten it to form a sector of a circle with radius l and central angle $\theta = 2\pi r/l$. We know that, in general, the area of a sector of a circle with radius l and angle θ is $\frac{1}{2}l^2\theta$ (see Exercise 67 in Section 6.2) and so in this case it is

$$A = \tfrac{1}{2}l^2\theta = \tfrac{1}{2}l^2\left(\frac{2\pi r}{l}\right) = \pi rl$$

Therefore, we define the lateral surface area of a cone to be $A = \pi rl$.

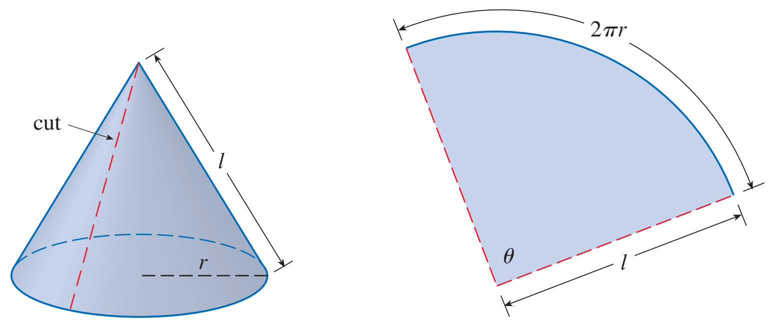


FIGURE 2

What about more complicated surfaces of revolution? If we follow the strategy we used with arc length, we can approximate the original curve by a polygon. When this polygon is rotated about an axis, it creates a simpler surface whose surface area approximates the actual surface area. By taking a limit, we can determine the exact surface area.

The approximating surface, then, consists of a number of *bands,* each formed by rotating a line segment about an axis. To find the surface area, each of these bands can be considered a portion of a circular cone, as shown in Figure 3. The area of the band (or frustum of a cone) with slant height l and upper and lower radii r_1 and r_2 is found by subtracting the areas of two cones:

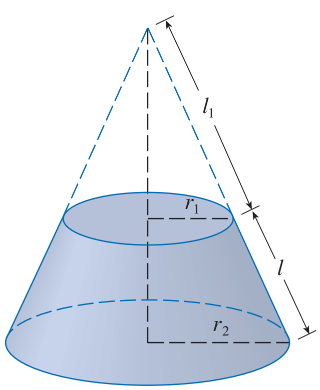


FIGURE 3

1 $$A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi[(r_2 - r_1)l_1 + r_2 l]$$

From similar triangles we have

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

which gives

$$r_2 l_1 = r_1 l_1 + r_1 l \qquad \text{or} \qquad (r_2 - r_1)l_1 = r_1 l$$

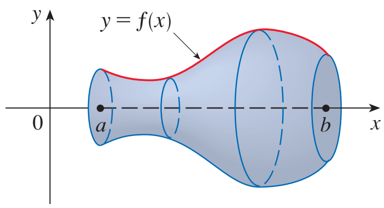
Putting this in Equation 1, we get

$$A = \pi(r_1 l + r_2 l)$$

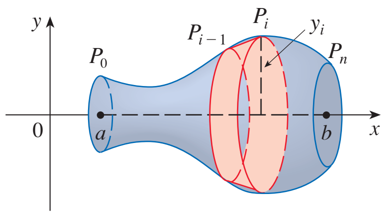
or

2 $$A = 2\pi rl$$

where $r = \frac{1}{2}(r_1 + r_2)$ is the average radius of the band.



(a) Surface of revolution



(b) Approximating band

FIGURE 4

Now we apply this formula to our strategy. Consider the surface shown in Figure 4, which is obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where f is positive and has a continuous derivative. In order to define its surface area, we divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx, as we did in determining arc length. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on the curve. The part of the surface between x_{i-1} and x_i is approximated by taking the line segment $P_{i-1}P_i$ and rotating it about the x-axis. The result is a band with slant height $l = |P_{i-1}P_i|$ and average radius $r = \frac{1}{2}(y_{i-1} + y_i)$ so, by Formula 2, its surface area is

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i|$$

As in the proof of Theorem 7.4.2, we have

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

where x_i^* is some number in $[x_{i-1}, x_i]$. When Δx is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$, since f is continuous. Therefore

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

and so an approximation to what we think of as the area of the complete surface of revolution is

3
$$\sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

This approximation appears to become better as $n \to \infty$ and, recognizing (3) as a Riemann sum for the function $g(x) = 2\pi f(x) \sqrt{1 + [f'(x)]^2}$, we have

$$\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

Therefore, in the case where f is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis as

4
$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

With the Leibniz notation for derivatives, this formula becomes

5
$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve is described as $x = g(y)$, $c \leqslant y \leqslant d$, then the formula for surface area becomes

6
$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

and both Formulas 5 and 6 can be summarized symbolically, using the notation for arc length given in Section 7.4, as

7 $$S = \int 2\pi y \, ds$$

For rotation about the y-axis, the surface area formula becomes

8 $$S = \int 2\pi x \, ds$$

where, as before, we can use either

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

These formulas can be remembered by thinking of $2\pi y$ or $2\pi x$ as the circumference of a circle traced out by the point (x, y) on the curve as it is rotated about the x-axis or y-axis, respectively (see Figure 5).

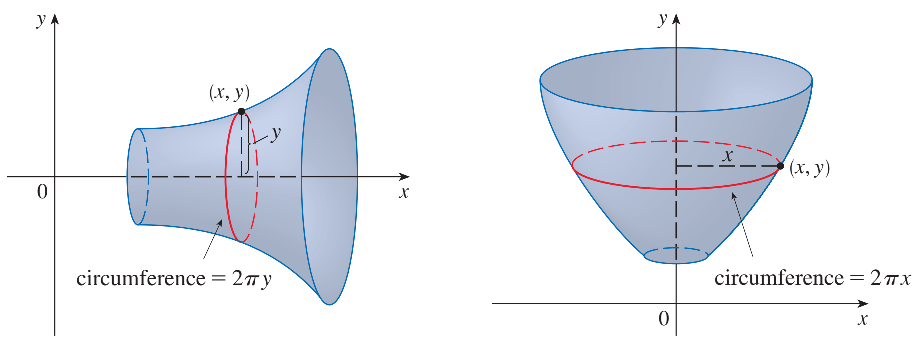


FIGURE 5 (a) Rotation about x-axis: $S = \int 2\pi y \, ds$ (b) Rotation about y-axis: $S = \int 2\pi x \, ds$

EXAMPLE 1 The curve $y = \sqrt{4 - x^2}$, $-1 \leqslant x \leqslant 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)

SOLUTION We have

$$\frac{dy}{dx} = \tfrac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

and so, by Formula 5, the surface area is

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \frac{2}{\sqrt{4 - x^2}}\, dx$$

$$= 4\pi \int_{-1}^{1} 1 \, dx = 4\pi(2) = 8\pi$$ ■

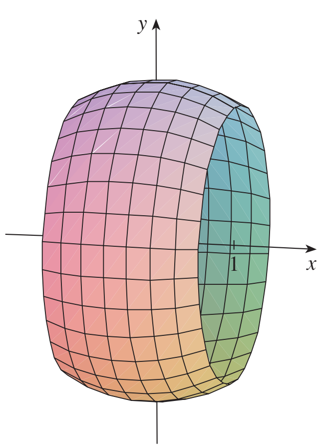


FIGURE 6

■■ Figure 6 shows the portion of the sphere whose surface area is computed in Example 1.

■■ Figure 7 shows the surface of revolution whose area is computed in Example 2.

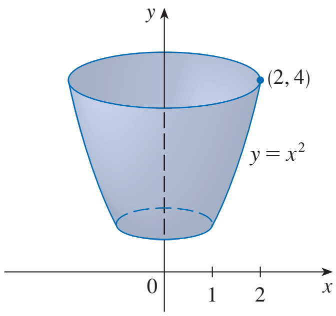


FIGURE 7

EXAMPLE 2 The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

SOLUTION 1 Using

$$y = x^2 \qquad \text{and} \qquad \frac{dy}{dx} = 2x$$

we have, from Formula 8,

$$\begin{aligned} S &= \int 2\pi x \, ds \\ &= \int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\ &= 2\pi \int_1^2 x \sqrt{1 + 4x^2} \, dx \end{aligned}$$

Substituting $u = 1 + 4x^2$, we have $du = 8x\,dx$. Remembering to change the limits of integration, we have

$$\begin{aligned} S &= \frac{\pi}{4} \int_5^{17} \sqrt{u} \, du = \frac{\pi}{4} \left[\tfrac{2}{3} u^{3/2}\right]_5^{17} \\ &= \frac{\pi}{6} \left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

■■ As a check on our answer to Example 2, notice from Figure 7 that the surface area should be close to that of a circular cylinder with the same height and radius halfway between the upper and lower radius of the surface: $2\pi(1.5)(3) \approx 28.27$. We computed that the surface area was

$$\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \approx 30.85$$

which seems reasonable. Alternatively, the surface area should be slightly larger than the area of a frustum of a cone with the same top and bottom edges. From Equation 2, this is $2\pi(1.5)(\sqrt{10}) \approx 29.80$.

SOLUTION 2 Using

$$x = \sqrt{y} \qquad \text{and} \qquad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

we have

$$\begin{aligned} S &= \int 2\pi x \, ds = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \\ &= 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}} \, dy = \pi \int_1^4 \sqrt{4y + 1} \, dy \\ &= \frac{\pi}{4} \int_5^{17} \sqrt{u} \, du \qquad \text{(where } u = 1 + 4y\text{)} \\ &= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \qquad \text{(as in Solution 1)} \end{aligned}$$

■■

EXAMPLE 3 Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leqslant x \leqslant 1$, about the x-axis.

■■ Another method: Use Formula 6 with $x = \ln y$.

SOLUTION Using Formula 5 with

$$y = e^x \qquad \text{and} \qquad \frac{dy}{dx} = e^x$$

we have

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx$$

$$= 2\pi \int_1^e \sqrt{1 + u^2}\, du \qquad \text{(where } u = e^x\text{)}$$

$$= 2\pi \int_{\pi/4}^{\alpha} \sec^3\theta\, d\theta \qquad \text{(where } u = \tan\theta \text{ and } \alpha = \tan^{-1}e\text{)}$$

■■ Or use Formula 21 in the Table of Integrals.

$$= 2\pi \cdot \tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\pi/4}^{\alpha} \qquad \text{(by Example 8 in Section 6.2)}$$

$$= \pi\Big[\sec\alpha\tan\alpha + \ln(\sec\alpha + \tan\alpha) - \sqrt{2} - \ln(\sqrt{2} + 1)\Big]$$

Since $\tan\alpha = e$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 1 + e^2$ and

$$S = \pi\Big[e\sqrt{1 + e^2} + \ln\big(e + \sqrt{1 + e^2}\big) - \sqrt{2} - \ln\big(\sqrt{2} + 1\big)\Big]$$

EXERCISES

A Click here for answers. S Click here for solutions.

1–4 ■ Set up, but do not evaluate, an integral for the area of the surface obtained by rotating the curve about the given axis.

1. $y = \ln x$, $1 \le x \le 3$; x-axis

2. $y = \sin^2 x$, $0 \le x \le \pi/2$; x-axis

3. $y = \sec x$, $0 \le x \le \pi/4$; y-axis

4. $y = e^x$, $1 \le y \le 2$; y-axis

5–12 ■ Find the area of the surface obtained by rotating the curve about the x-axis.

5. $y = x^3$, $0 \le x \le 2$

6. $9x = y^2 + 18$, $2 \le x \le 6$

7. $y = \sqrt{x}$, $4 \le x \le 9$

8. $y = \cos 2x$, $0 \le x \le \pi/6$

9. $y = \cosh x$, $0 \le x \le 1$

10. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $\tfrac{1}{2} \le x \le 1$

11. $x = \tfrac{1}{3}(y^2 + 2)^{3/2}$, $1 \le y \le 2$

12. $x = 1 + 2y^2$, $1 \le y \le 2$

13–16 ■ The given curve is rotated about the y-axis. Find the area of the resulting surface.

13. $y = \sqrt[3]{x}$, $1 \le y \le 2$

14. $y = 1 - x^2$, $0 \le x \le 1$

15. $x = \sqrt{a^2 - y^2}$, $0 \le y \le a/2$

16. $x = a\cosh(y/a)$, $-a \le y \le a$

17–20 ■ Use Simpson's Rule with $n = 10$ to approximate the area of the surface obtained by rotating the curve about the x-axis. Compare your answer with the value of the integral produced by your calculator.

17. $y = \ln x$, $1 \le x \le 3$

18. $y = x + \sqrt{x}$, $1 \le x \le 2$

19. $y = \sec x$, $0 \le x \le \pi/3$

20. $y = \sqrt{1 + e^x}$, $0 \le x \le 1$

CAS **21–22** ■ Use either a CAS or a table of integrals to find the exact area of the surface obtained by rotating the given curve about the x-axis.

21. $y = 1/x$, $1 \le x \le 2$

22. $y = \sqrt{x^2 + 1}$, $0 \le x \le 3$

CAS **23–24** ■ Use a CAS to find the exact area of the surface obtained by rotating the curve about the y-axis. If your CAS has trouble evaluating the integral, express the surface area as an integral in the other variable.

23. $y = x^3$, $0 \le y \le 1$

24. $y = \ln(x + 1)$, $0 \le x \le 1$

25. (a) If $a > 0$, find the area of the surface generated by rotating the loop of the curve $3ay^2 = x(a - x)^2$ about the x-axis.
(b) Find the surface area if the loop is rotated about the y-axis.

26. A group of engineers is building a parabolic satellite dish whose shape will be formed by rotating the curve $y = ax^2$ about the y-axis. If the dish is to have a 10-ft diameter and a maximum depth of 2 ft, find the value of a and the surface area of the dish.

27. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b$$

is rotated about the x-axis to form a surface called an *ellipsoid*. Find the surface area of this ellipsoid.

28. Find the surface area of the torus in Exercise 41 in Section 7.2.

29. If the curve $y = f(x)$, $a \leqslant x \leqslant b$, is rotated about the horizontal line $y = c$, where $f(x) \leqslant c$, find a formula for the area of the resulting surface.

CAS **30.** Use the result of Exercise 29 to set up an integral to find the area of the surface generated by rotating the curve $y = \sqrt{x}$, $0 \leqslant x \leqslant 4$, about the line $y = 4$. Then use a CAS to evaluate the integral.

31. Find the area of the surface obtained by rotating the circle $x^2 + y^2 = r^2$ about the line $y = r$.

32. Show that the surface area of a zone of a sphere that lies between two parallel planes is $S = \pi dh$, where d is the diameter of the sphere and h is the distance between the planes. (Notice that S depends only on the distance between the planes and not on their location, provided that both planes intersect the sphere.)

33. Formula 4 is valid only when $f(x) \geqslant 0$. Show that when $f(x)$ is not necessarily positive, the formula for surface area becomes

$$S = \int_a^b 2\pi \left| f(x) \right| \sqrt{1 + [f'(x)]^2}\, dx$$

34. Let L be the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, where f is positive and has a continuous derivative. Let S_f be the surface area generated by rotating the curve about the x-axis. If c is a positive constant, define $g(x) = f(x) + c$ and let S_g be the corresponding surface area generated by the curve $y = g(x)$, $a \leqslant x \leqslant b$. Express S_g in terms of S_f and L.

ANSWERS

S Click here for solutions.

1. $\int_1^3 2\pi \ln x \sqrt{1 + (1/x)^2}\, dx$

3. $\int_0^{\pi/4} 2\pi x \sqrt{1 + (\sec x \tan x)^2}\, dx$

5. $\pi(145\sqrt{145} - 1)/27$ **7.** $\pi(37\sqrt{37} - 17\sqrt{17})/6$

9. $\pi[1 + \frac{1}{4}(e^2 - e^{-2})]$ **11.** $21\pi/2$

13. $\pi(145\sqrt{145} - 10\sqrt{10})/27$ **15.** πa^2

17. 9.023754 **19.** 13.527296

21. $(\pi/4)[4\ln(\sqrt{17} + 4) - 4\ln(\sqrt{2} + 1) - \sqrt{17} + 4\sqrt{2}]$

23. $(\pi/6)[\ln(\sqrt{10} + 3) + 3\sqrt{10}]$

25. (a) $\pi a^2/3$ (b) $56\pi\sqrt{3}a^2/45$

27. $2\pi[b^2 + a^2 b \sin^{-1}(\sqrt{a^2 - b^2}/a)/\sqrt{a^2 - b^2}]$

29. $\int_a^b 2\pi[c - f(x)]\sqrt{1 + [f'(x)]^2}\, dx$ **31.** $4\pi^2 r^2$

SOLUTIONS

1. $y = \ln x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (1/x)^2}\,dx \;\Rightarrow\; S = \int_1^3 2\pi(\ln x)\sqrt{1 + (1/x)^2}\,dx$ [by (7)]

3. $y = \sec x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (\sec x \tan x)^2}\,dx \;\Rightarrow$

$S = \int_0^{\pi/4} 2\pi x\,\sqrt{1 + (\sec x \tan x)^2}\,dx$ [by (8)]

5. $y = x^3 \;\Rightarrow\; y' = 3x^2$. So

$$S = \int_0^2 2\pi y\,\sqrt{1 + (y')^2}\,dx = 2\pi \int_0^2 x^3\sqrt{1 + 9x^4}\,dx \qquad [u = 1 + 9x^4,\ du = 36x^3\,dx]$$
$$= \tfrac{2\pi}{36}\int_1^{145}\sqrt{u}\,du = \tfrac{\pi}{18}\left[\tfrac{2}{3}u^{3/2}\right]_1^{145} = \tfrac{\pi}{27}\left(145\sqrt{145} - 1\right)$$

7. $y = \sqrt{x} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \left[1/(2\sqrt{x}\,)\right]^2 = 1 + 1/(4x)$. So

$$S = \int_4^9 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_4^9 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx = 2\pi\int_4^9 \sqrt{x + \tfrac{1}{4}}\,dx$$
$$= 2\pi\left[\tfrac{2}{3}\left(x + \tfrac{1}{4}\right)^{3/2}\right]_4^9 = \tfrac{4\pi}{3}\left[\tfrac{1}{8}(4x + 1)^{3/2}\right]_4^9 = \tfrac{\pi}{6}\left(37\sqrt{37} - 17\sqrt{17}\right)$$

9. $y = \cosh x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \sinh^2 x = \cosh^2 x$. So

$$S = 2\pi\int_0^1 \cosh x \cosh x\,dx = 2\pi\int_0^1 \tfrac{1}{2}(1 + \cosh 2x)\,dx = \pi\left[x + \tfrac{1}{2}\sinh 2x\right]_0^1$$
$$= \pi\left(1 + \tfrac{1}{2}\sinh 2\right) \quad\text{or}\quad \pi\left[1 + \tfrac{1}{4}\left(e^2 - e^{-2}\right)\right]$$

11. $x = \frac{1}{3}\left(y^2 + 2\right)^{3/2} \;\Rightarrow\; dx/dy = \frac{1}{2}\left(y^2 + 2\right)^{1/2}(2y) = y\sqrt{y^2 + 2} \;\Rightarrow$

$1 + (dx/dy)^2 = 1 + y^2\left(y^2 + 2\right) = \left(y^2 + 1\right)^2$. So

$$S = 2\pi\int_1^2 y\left(y^2 + 1\right)dy = 2\pi\left[\tfrac{1}{4}y^4 + \tfrac{1}{2}y^2\right]_1^2 = 2\pi\left(4 + 2 - \tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{21\pi}{2}$$

13. $y = \sqrt[3]{x} \;\Rightarrow\; x = y^3 \;\Rightarrow\; 1 + (dx/dy)^2 = 1 + 9y^4$. So

$$S = 2\pi\int_1^2 x\sqrt{1 + (dx/dy)^2}\,dy = 2\pi\int_1^2 y^3\sqrt{1 + 9y^4}\,dy = \tfrac{2\pi}{36}\int_1^2 \sqrt{1 + 9y^4}\,36y^3\,dy$$
$$= \tfrac{\pi}{18}\left[\tfrac{2}{3}\left(1 + 9y^4\right)^{3/2}\right]_1^2 = \tfrac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)$$

15. $x = \sqrt{a^2 - y^2} \;\Rightarrow\; dx/dy = \frac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = -y/\sqrt{a^2 - y^2} \;\Rightarrow$

$$1 + (dx/dy)^2 = 1 + \frac{y^2}{a^2 - y^2} = \frac{a^2 - y^2}{a^2 - y^2} + \frac{y^2}{a^2 - y^2} = \frac{a^2}{a^2 - y^2} \;\Rightarrow$$

$S = \displaystyle\int_0^{a/2} 2\pi\sqrt{a^2 - y^2}\frac{a}{\sqrt{a^2 - y^2}}\,dy = 2\pi\int_0^{a/2} a\,dy = 2\pi a\left[y\right]_0^{a/2} = 2\pi a\left(\frac{a}{2} - 0\right) = \pi a^2$. Note that this is $\frac{1}{4}$ the surface area of a sphere of radius a, and the length of the interval $y = 0$ to $y = a/2$ is $\frac{1}{4}$ the length of the interval $y = -a$ to $y = a$.

17. $y = \ln x \;\Rightarrow\; dy/dx = 1/x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + 1/x^2 \;\Rightarrow\; S = \int_1^3 2\pi \ln x \sqrt{1 + 1/x^2}\, dx.$

Let $f(x) = \ln x \sqrt{1 + 1/x^2}$. Since $n = 10$, $\Delta x = \frac{3-1}{10} = \frac{1}{5}$. Then

$S \approx S_{10} = 2\pi \cdot \frac{1/5}{3} [f(1) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3)] \approx 9.023754.$

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

19. $y = \sec x \;\Rightarrow\; dy/dx = \sec x \tan x \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \sec^2 x \tan^2 x \;\Rightarrow$

$S = \int_0^{\pi/3} 2\pi \sec x \sqrt{1 + \sec^2 x \tan^2 x}\, dx$. Let $f(x) = \sec x \sqrt{1 + \sec^2 x \tan^2 x}$.

Since $n = 10$, $\Delta x = \dfrac{\pi/3 - 0}{10} = \dfrac{\pi}{30}$. Then

$$S \approx S_{10} = 2\pi \cdot \frac{\pi/30}{3}\left[f(0) + 4f\left(\frac{\pi}{30}\right) + 2f\left(\frac{2\pi}{30}\right) + \cdots + 2f\left(\frac{8\pi}{30}\right) + 4f\left(\frac{9\pi}{30}\right) + f\left(\frac{\pi}{3}\right)\right] \approx 13.527296.$$

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

21. $y = 1/x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + (-1/x^2)^2}\, dx = \sqrt{1 + 1/x^4}\, dx \;\Rightarrow$

$$S = \int_1^2 2\pi \cdot \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx = 2\pi \int_1^2 \frac{\sqrt{x^4+1}}{x^3}\, dx = 2\pi \int_1^4 \frac{\sqrt{u^2+1}}{u^2}\left(\tfrac{1}{2}\, du\right) \qquad [u = x^2,\ du = 2x\, dx]$$

$$= \pi \int_1^4 \frac{\sqrt{1+u^2}}{u^2}\, du \overset{24}{=} \pi\left[-\frac{\sqrt{1+u^2}}{u} + \ln\left(u + \sqrt{1+u^2}\right)\right]_1^4$$

$$= \pi\left[-\tfrac{\sqrt{17}}{4} + \ln\left(4 + \sqrt{17}\right) + \tfrac{\sqrt{2}}{1} - \ln\left(1 + \sqrt{2}\right)\right] = \pi\left[\sqrt{2} - \tfrac{\sqrt{17}}{4} + \ln\left(\tfrac{4+\sqrt{17}}{1+\sqrt{2}}\right)\right]$$

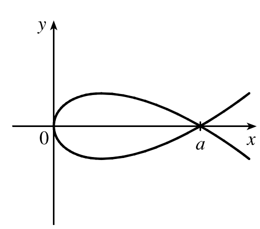
23. $y = x^3$ and $0 \le y \le 1 \;\Rightarrow\; y' = 3x^2$ and $0 \le x \le 1$.

$$S = \int_0^1 2\pi x \sqrt{1 + (3x^2)^2}\, dx = 2\pi \int_0^3 \sqrt{1+u^2}\, \tfrac{1}{6}\, du \qquad [u = 3x^2,\ du = 6x\, dx]$$

$$= \tfrac{\pi}{3}\int_0^3 \sqrt{1+u^2}\, du \overset{21}{=} \text{[or use CAS]}\ \tfrac{\pi}{3}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^3$$

$$= \tfrac{\pi}{3}\left[\tfrac{3}{2}\sqrt{10} + \tfrac{1}{2}\ln\left(3 + \sqrt{10}\right)\right] = \tfrac{\pi}{6}\left[3\sqrt{10} + \ln\left(3 + \sqrt{10}\right)\right]$$

25. Since $a > 0$, the curve $3ay^2 = x(a-x)^2$ only has points with $x \ge 0$. ($3ay^2 \ge 0 \;\Rightarrow\; x(a-x)^2 \ge 0 \;\Rightarrow\; x \ge 0$.) The curve is symmetric about the x-axis (since the equation is unchanged when y is replaced by $-y$). $y = 0$ when $x = 0$ or a, so the curve's loop extends from $x = 0$ to $x = a$.



$$\frac{d}{dx}(3ay^2) = \frac{d}{dx}[x(a-x)^2] \;\Rightarrow\; 6ay\frac{dy}{dx} = x \cdot 2(a-x)(-1) + (a-x)^2 \;\Rightarrow\; \frac{dy}{dx} = \frac{(a-x)[-2x + a - x]}{6ay}$$

$$\Rightarrow\; \left(\frac{dy}{dx}\right)^2 = \frac{(a-x)^2(a-3x)^2}{36a^2y^2} = \frac{(a-x)^2(a-3x)^2}{36a^2} \cdot \frac{3a}{x(a-x)^2}\ \left[\begin{array}{c}\text{the last fraction}\\ \text{is } 1/y^2\end{array}\right] = \frac{(a-3x)^2}{12ax} \;\Rightarrow$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{a^2-6ax+9x^2}{12ax}=\frac{12ax}{12ax}+\frac{a^2-6ax+9x^2}{12ax}=\frac{a^2+6ax+9x^2}{12ax}=\frac{(a+3x)^2}{12ax}\text{ for } x\neq 0.$$

(a) $S=\int_{x=0}^{a}2\pi y\,ds=2\pi\int_0^a\frac{\sqrt{x}(a-x)}{\sqrt{3a}}\cdot\frac{a+3x}{\sqrt{12ax}}\,dx=2\pi\int_0^a\frac{(a-x)(a+3x)}{6a}\,dx$

$$=\frac{\pi}{3a}\int_0^a(a^2+2ax-3x^2)\,dx=\frac{\pi}{3a}\left[a^2x+ax^2-x^3\right]_0^a=\frac{\pi}{3a}(a^3+a^3-a^3)=\frac{\pi}{3a}\cdot a^3=\frac{\pi a^2}{3}.$$

Note that we have rotated the top half of the loop about the x-axis. This generates the full surface.

(b) We must rotate the full loop about the y-axis, so we get double the area obtained by rotating the top half of the loop:

$$S=2\cdot 2\pi\int_{x=0}^{a}x\,ds=4\pi\int_0^a x\,\frac{a+3x}{\sqrt{12ax}}\,dx=\frac{4\pi}{2\sqrt{3a}}\int_0^a x^{1/2}(a+3x)\,dx$$

$$=\frac{2\pi}{\sqrt{3a}}\int_0^a(ax^{1/2}+3x^{3/2})\,dx=\frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}ax^{3/2}+\frac{6}{5}x^{5/2}\right]_0^a=\frac{2\pi\sqrt{3}}{3\sqrt{a}}\left(\frac{2}{3}a^{5/2}+\frac{6}{5}a^{5/2}\right)$$

$$=\frac{2\pi\sqrt{3}}{3}\left(\frac{2}{3}+\frac{6}{5}\right)a^2=\frac{2\pi\sqrt{3}}{3}\left(\frac{28}{15}\right)a^2=\frac{56\pi\sqrt{3}\,a^2}{45}$$

27. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\quad\Rightarrow\quad\frac{y\,(dy/dx)}{b^2}=-\frac{x}{a^2}\quad\Rightarrow\quad\frac{dy}{dx}=-\frac{b^2x}{a^2y}\quad\Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{b^4x^2}{a^4y^2}=\frac{b^4x^2+a^4y^2}{a^4y^2}=\frac{b^4x^2+a^4b^2\left(1-x^2/a^2\right)}{a^4b^2\left(1-x^2/a^2\right)}=\frac{a^4b^2+b^4x^2-a^2b^2x^2}{a^4b^2-a^2b^2x^2}$$

$$=\frac{a^4+b^2x^2-a^2x^2}{a^4-a^2x^2}=\frac{a^4-\left(a^2-b^2\right)x^2}{a^2(a^2-x^2)}$$

The ellipsoid's surface area is twice the area generated by rotating the first quadrant portion of the ellipse about the x-axis. Thus,

$$S=2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\pi\int_0^a\frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx$$

$$=\frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx=\frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}}\sqrt{a^4-u^2}\,\frac{du}{\sqrt{a^2-b^2}}\qquad\left[u=\sqrt{a^2-b^2}\,x\right]$$

$$\overset{30}{=}\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{a^4-u^2}+\frac{a^4}{2}\sin^{-1}\frac{u}{a^2}\right]_0^{a\sqrt{a^2-b^2}}$$

$$=\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{a\sqrt{a^2-b^2}}{2}\sqrt{a^4-a^2(a^2-b^2)}+\frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]=2\pi\left[b^2+\frac{a^2b\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]$$

29. The analogue of $f(x_i^*)$ in the derivation of (4) is now $c-f(x_i^*)$, so

$$S=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi[c-f(x_i^*)]\sqrt{1+[f'(x_i^*)]^2}\,\Delta x=\int_a^b 2\pi[c-f(x)]\sqrt{1+[f'(x)]^2}\,dx.$$

31. For the upper semicircle, $f(x) = \sqrt{r^2 - x^2}$, $f'(x) = -x/\sqrt{r^2 - x^2}$. The surface area generated is

$$S_1 = \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2 - x^2}\right)\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = 4\pi\int_0^r \left(r - \sqrt{r^2 - x^2}\right)\frac{r}{\sqrt{r^2 - x^2}}\,dx$$

$$= 4\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} - r\right) dx$$

For the lower semicircle, $f(x) = -\sqrt{r^2 - x^2}$ and $f'(x) = \dfrac{x}{\sqrt{r^2 - x^2}}$, so $S_2 = 4\pi \displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} + r\right) dx$.

Thus, the total area is $S = S_1 + S_2 = 8\pi \displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}}\right) dx = 8\pi\left[r^2 \sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 8\pi r^2\left(\frac{\pi}{2}\right) = 4\pi^2 r^2$.

33. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi\,\dfrac{y_{i-1} + y_i}{2}\,|P_{i-1}P_i|$, the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i = f(x_i) \approx f(x_i^*)$ and $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ must be replaced by $y_i = |f(x_i)| \approx |f(x_i^*)|$ and $y_{i-1} = |f(x_{i-1})| \approx |f(x_i^*)|$. Thus, $2\pi\,\dfrac{y_{i-1} + y_i}{2}\,|P_{i-1}P_i| \approx 2\pi\,|f(x_i^*)|\,\sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$. Continuing with the rest of the derivation as before, we obtain $S = \int_a^b 2\pi\,|f(x)|\,\sqrt{1 + [f'(x)]^2}\,dx$.