

## LINEAR DIFFERENTIAL EQUATIONS

A first-order **linear** differential equation is one that can be put into the form

$$\boxed{1} \quad \frac{dy}{dx} + P(x)y = Q(x)$$

where  $P$  and  $Q$  are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

An example of a linear equation is  $xy' + y = 2x$  because, for  $x \neq 0$ , it can be written in the form

$$\boxed{2} \quad y' + \frac{1}{x}y = 2$$

Notice that this differential equation is not separable because it's impossible to factor the expression for  $y'$  as a function of  $x$  times a function of  $y$ . But we can still solve the equation by noticing, by the Product Rule, that

$$xy' + y = (xy)'$$

and so we can rewrite the equation as

$$(xy)' = 2x$$

If we now integrate both sides of this equation, we get

$$xy = x^2 + C \quad \text{or} \quad y = x + \frac{C}{x}$$

If we had been given the differential equation in the form of Equation 2, we would have had to take the preliminary step of multiplying each side of the equation by  $x$ .

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function  $I(x)$  called an *integrating factor*. We try to find  $I$  so that the left side of Equation 1, when multiplied by  $I(x)$ , becomes the derivative of the product  $I(x)y$ :

$$\boxed{3} \quad I(x)(y' + P(x)y) = (I(x)y)'$$

If we can find such a function  $I$ , then Equation 1 becomes

$$(I(x)y)' = I(x)Q(x)$$

Integrating both sides, we would have

$$I(x)y = \int I(x)Q(x) dx + C$$

so the solution would be

$$\boxed{4} \quad y(x) = \frac{1}{I(x)} \left[ \int I(x)Q(x) dx + C \right]$$

To find such an  $I$ , we expand Equation 3 and cancel terms:

$$I(x)y' + I(x)P(x)y = (I(x)y)' = I'(x)y + I(x)y'$$

$$I(x)P(x) = I'(x)$$

This is a separable differential equation for  $I$ , which we solve as follows:

$$\int \frac{dI}{I} = \int P(x) dx$$

$$\ln |I| = \int P(x) dx$$

$$I = Ae^{\int P(x) dx}$$

where  $A = \pm e^C$ . We are looking for a particular integrating factor, not the most general one, so we take  $A = 1$  and use

$$\boxed{5} \quad I(x) = e^{\int P(x) dx}$$

Thus, a formula for the general solution to Equation 1 is provided by Equation 4, where  $I$  is given by Equation 5. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

To solve the linear differential equation  $y' + P(x)y = Q(x)$ , multiply both sides by the **integrating factor**  $I(x) = e^{\int P(x) dx}$  and integrate both sides.

**EXAMPLE 1** Solve the differential equation  $\frac{dy}{dx} + 3x^2y = 6x^2$ .

**SOLUTION** The given equation is linear since it has the form of Equation 1 with  $P(x) = 3x^2$  and  $Q(x) = 6x^2$ . An integrating factor is

$$I(x) = e^{\int 3x^2 dx} = e^{x^3}$$

Multiplying both sides of the differential equation by  $e^{x^3}$ , we get

$$e^{x^3} \frac{dy}{dx} + 3x^2 e^{x^3} y = 6x^2 e^{x^3}$$

or

$$\frac{d}{dx} (e^{x^3} y) = 6x^2 e^{x^3}$$

Integrating both sides, we have

$$e^{x^3} y = \int 6x^2 e^{x^3} dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

■ ■ Figure 1 shows the graphs of several members of the family of solutions in Example 1. Notice that they all approach 2 as  $x \rightarrow \infty$ .

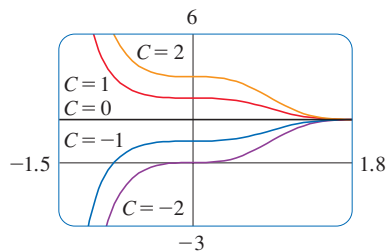


FIGURE 1

**EXAMPLE 2** Find the solution of the initial-value problem

$$x^2y' + xy = 1 \quad x > 0 \quad y(1) = 2$$

**SOLUTION** We must first divide both sides by the coefficient of  $y'$  to put the differential equation into standard form:

$$\boxed{6} \quad y' + \frac{1}{x}y = \frac{1}{x^2} \quad x > 0$$

The integrating factor is

$$I(x) = e^{\int (1/x) dx} = e^{\ln x} = x$$

Multiplication of Equation 6 by  $x$  gives

$$xy' + y = \frac{1}{x} \quad \text{or} \quad (xy)' = \frac{1}{x}$$

Then

$$xy = \int \frac{1}{x} dx = \ln x + C$$

and so

$$y = \frac{\ln x + C}{x}$$

Since  $y(1) = 2$ , we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Therefore, the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}$$

■ ■ The solution of the initial-value problem in Example 2 is shown in Figure 2.

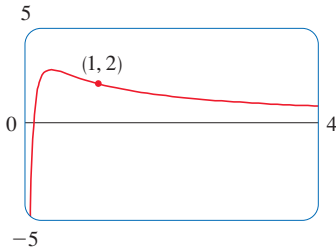


FIGURE 2

■ ■ Even though the solutions of the differential equation in Example 3 are expressed in terms of an integral, they can still be graphed by a computer algebra system (Figure 3).

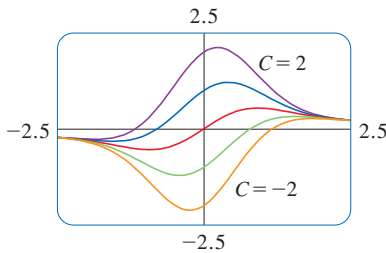


FIGURE 3

**EXAMPLE 3** Solve  $y' + 2xy = 1$ .

**SOLUTION** The given equation is in the standard form for a linear equation. Multiplying by the integrating factor

$$e^{\int 2x dx} = e^{x^2}$$

we get

$$e^{x^2}y' + 2xe^{x^2}y = e^{x^2}$$

or

$$(e^{x^2}y)' = e^{x^2}$$

Therefore

$$e^{x^2}y = \int e^{x^2} dx + C$$

Recall from Section 6.4 that  $\int e^{x^2} dx$  can't be expressed in terms of elementary functions. Nonetheless, it's a perfectly good function and we can leave the answer as

$$y = e^{-x^2} \int e^{x^2} dx + Ce^{-x^2}$$

Another way of writing the solution is

$$y = e^{-x^2} \int_0^x e^{t^2} dt + Ce^{-x^2}$$

(Any number can be chosen for the lower limit of integration.)

### APPLICATION TO ELECTRIC CIRCUITS

Let's consider the simple electric circuit shown in Figure 4: An electromotive force (usually a battery or generator) produces a voltage of  $E(t)$  volts (V) and a current of  $I(t)$  amperes (A) at time  $t$ . The circuit also contains a resistor with a resistance of  $R$  ohms ( $\Omega$ ) and an inductor with an inductance of  $L$  henries (H).

Ohm's Law gives the drop in voltage due to the resistor as  $RI$ . The voltage drop due to the inductor is  $L(dI/dt)$ . One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage  $E(t)$ . Thus, we have

$$\boxed{7} \quad L \frac{dI}{dt} + RI = E(t)$$

which is a first-order linear differential equation. The solution gives the current  $I$  at time  $t$ .

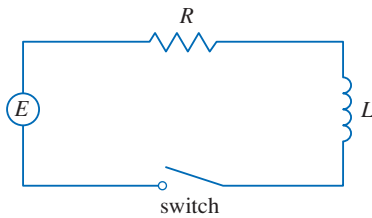


FIGURE 4

■ ■ The differential equation in Example 4 is both linear and separable, so an alternative method is to solve it as a separable equation. If we replace the battery by a generator, however, we get an equation that is linear but not separable (Example 5).

■ ■ Figure 5 shows how the current in Example 4 approaches its limiting value.

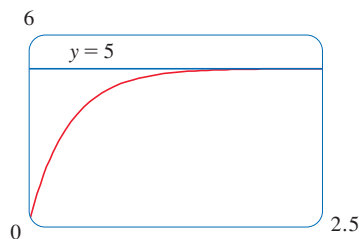


FIGURE 5

■ ■ Figure 6 shows the graph of the current when the battery is replaced by a generator.

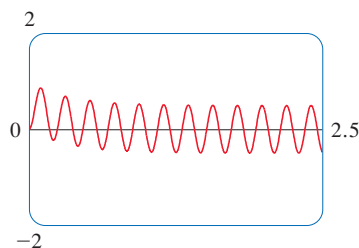


FIGURE 6

**EXAMPLE 4** Suppose that in the simple circuit of Figure 4 the resistance is  $12\ \Omega$  and the inductance is  $4\ \text{H}$ . If a battery gives a constant voltage of  $60\ \text{V}$  and the switch is closed when  $t = 0$  so the current starts with  $I(0) = 0$ , find (a)  $I(t)$ , (b) the current after  $1\ \text{s}$ , and (c) the limiting value of the current.

**SOLUTION**

(a) If we put  $L = 4$ ,  $R = 12$ , and  $E(t) = 60$  in Equation 7, we obtain the initial-value problem

$$4 \frac{dI}{dt} + 12I = 60 \quad I(0) = 0$$

or 
$$\frac{dI}{dt} + 3I = 15 \quad I(0) = 0$$

Multiplying by the integrating factor  $e^{\int 3 dt} = e^{3t}$ , we get

$$e^{3t} \frac{dI}{dt} + 3e^{3t}I = 15e^{3t}$$

$$\frac{d}{dt}(e^{3t}I) = 15e^{3t}$$

$$e^{3t}I = \int 15e^{3t} dt = 5e^{3t} + C$$

$$I(t) = 5 + Ce^{-3t}$$

Since  $I(0) = 0$ , we have  $5 + C = 0$ , so  $C = -5$  and

$$I(t) = 5(1 - e^{-3t})$$

(b) After 1 second the current is

$$I(1) = 5(1 - e^{-3}) \approx 4.75\ \text{A}$$

(c)

$$\begin{aligned} \lim_{t \rightarrow \infty} I(t) &= \lim_{t \rightarrow \infty} 5(1 - e^{-3t}) \\ &= 5 - 5 \lim_{t \rightarrow \infty} e^{-3t} \\ &= 5 - 0 = 5 \end{aligned}$$

**EXAMPLE 5** Suppose that the resistance and inductance remain as in Example 4 but, instead of the battery, we use a generator that produces a variable voltage of  $E(t) = 60 \sin 30t$  volts. Find  $I(t)$ .

**SOLUTION** This time the differential equation becomes

$$4 \frac{dI}{dt} + 12I = 60 \sin 30t \quad \text{or} \quad \frac{dI}{dt} + 3I = 15 \sin 30t$$

The same integrating factor  $e^{3t}$  gives

$$\frac{d}{dt}(e^{3t}I) = e^{3t} \frac{dI}{dt} + 3e^{3t}I = 15e^{3t} \sin 30t$$

Using Formula 98 in the Table of Integrals, we have

$$e^{3t}I = \int 15e^{3t} \sin 30t dt = 15 \frac{e^{3t}}{909} (3 \sin 30t - 30 \cos 30t) + C$$

$$I = \frac{5}{101} (\sin 30t - 10 \cos 30t) + Ce^{-3t}$$

Since  $I(0) = 0$ , we get

$$-\frac{50}{101} + C = 0$$

so

$$I(t) = \frac{5}{101}(\sin 30t - 10 \cos 30t) + \frac{50}{101}e^{-3t}$$

**EXERCISES**

**A** Click here for answers.

**S** Click here for solutions.

**1–4** ■ Determine whether the differential equation is linear.

- 1.  $y' + e^x y = x^2 y^2$
- 2.  $y + \sin x = x^3 y'$
- 3.  $xy' + \ln x - x^2 y = 0$
- 4.  $y' + \cos y = \tan x$

**5–14** ■ Solve the differential equation.

- 5.  $y' + 2y = 2e^x$
- 6.  $y' = x + 5y$
- 7.  $xy' - 2y = x^2$
- 8.  $x^2 y' + 2xy = \cos^2 x$
- 9.  $xy' + y = \sqrt{x}$
- 10.  $1 + xy = xy'$

11.  $\frac{dy}{dx} + 2xy = x^2$

12.  $\frac{dy}{dx} = x \sin 2x + y \tan x, \quad -\pi/2 < x < \pi/2$

13.  $(1 + t) \frac{du}{dt} + u = 1 + t, \quad t > 0$

14.  $t \ln t \frac{dr}{dt} + r = te^t$

**15–20** ■ Solve the initial-value problem.

- 15.  $y' = x + y, \quad y(0) = 2$
- 16.  $t \frac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(1) = 0$
- 17.  $\frac{dv}{dt} - 2tv = 3t^2 e^{t^2}, \quad v(0) = 5$
- 18.  $2xy' + y = 6x, \quad x > 0, \quad y(4) = 20$
- 19.  $xy' = y + x^2 \sin x, \quad y(\pi) = 0$
- 20.  $x \frac{dy}{dx} - \frac{y}{x+1} = x, \quad y(1) = 0, \quad x > 0$

**21–22** ■ Solve the differential equation and use a graphing calculator or computer to graph several members of the family of solutions. How does the solution curve change as  $C$  varies?

- 21.  $xy' + y = x \cos x, \quad x > 0$
- 22.  $y' + (\cos x)y = \cos x$

**23. A Bernoulli differential equation** (named after James Bernoulli) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Observe that, if  $n = 0$  or  $1$ , the Bernoulli equation is linear. For other values of  $n$ , show that the substitution  $u = y^{1-n}$  transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x)$$

**24–26** ■ Use the method of Exercise 23 to solve the differential equation.

24.  $xy' + y = -xy^2$

25.  $y' + \frac{2}{x}y = \frac{y^3}{x^2}$

26.  $y' + y = xy^3$

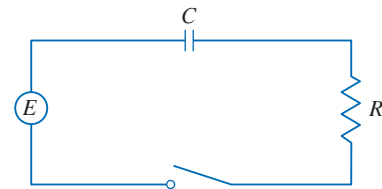
**27.** In the circuit shown in Figure 4, a battery supplies a constant voltage of 40 V, the inductance is 2 H, the resistance is 10  $\Omega$ , and  $I(0) = 0$ .

- (a) Find  $I(t)$ .
- (b) Find the current after 0.1 s.

**28.** In the circuit shown in Figure 4, a generator supplies a voltage of  $E(t) = 40 \sin 60t$  volts, the inductance is 1 H, the resistance is 20  $\Omega$ , and  $I(0) = 1$  A.

- (a) Find  $I(t)$ .
- (b) Find the current after 0.1 s.
- (c) Use a graphing device to draw the graph of the current function.

**29.** The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of  $C$  farads (F), and a resistor with a resistance of  $R$  ohms ( $\Omega$ ). The voltage drop across the



capacitor is  $Q/C$ , where  $Q$  is the charge (in coulombs), so in this case Kirchhoff's Law gives

$$RI + \frac{Q}{C} = E(t)$$

But  $I = dQ/dt$ , so we have

$$R \frac{dQ}{dt} + \frac{1}{C} Q = E(t)$$

Suppose the resistance is 5  $\Omega$ , the capacitance is 0.05 F, a battery gives a constant voltage of 60 V, and the initial charge is  $Q(0) = 0$  C. Find the charge and the current at time  $t$ .

30. In the circuit of Exercise 29,  $R = 2 \Omega$ ,  $C = 0.01 \text{ F}$ ,  $Q(0) = 0$ , and  $E(t) = 10 \sin 60t$ . Find the charge and the current at time  $t$ .
31. Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function  $P(t)$ , the performance of someone learning a skill as a function of the training time  $t$ . The derivative  $dP/dt$  represents the rate at which performance improves.
- (a) When do you think  $P$  increases most rapidly? What happens to  $dP/dt$  as  $t$  increases? Explain.
- (b) If  $M$  is the maximum level of performance of which the learner is capable, explain why the differential equation

$$\frac{dP}{dt} = k(M - P) \quad k \text{ a positive constant}$$

is a reasonable model for learning.

- (c) Solve the differential equation in part (b) as a linear differential equation and use your solution to graph the learning curve.
32. Two new workers were hired for an assembly line. Jim processed 25 units during the first hour and 45 units during the second hour. Mark processed 35 units during the first hour and 50 units the second hour. Using the model of Exercise 31 and assuming that  $P(0) = 0$ , estimate the maximum number of units per hour that each worker is capable of processing.
33. In Section 7.6 we looked at mixing problems in which the volume of fluid remained constant and saw that such problems give rise to separable equations. (See Example 5 in that section.) If the rates of flow into and out of the system are different, then the volume is not constant and the resulting differential equation is linear but not separable.

A tank contains 100 L of water. A solution with a salt concentration of 0.4 kg/L is added at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at a rate

of 3 L/min. If  $y(t)$  is the amount of salt (in kilograms) after  $t$  minutes, show that  $y$  satisfies the differential equation

$$\frac{dy}{dt} = 2 - \frac{3y}{100 + 2t}$$

Solve this equation and find the concentration after 20 minutes.

34. A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of 0.05 g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s. Find the amount of chlorine in the tank as a function of time.
35. An object with mass  $m$  is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If  $s(t)$  is the distance dropped after  $t$  seconds, then the speed is  $v = s'(t)$  and the acceleration is  $a = v'(t)$ . If  $g$  is the acceleration due to gravity, then the downward force on the object is  $mg - cv$ , where  $c$  is a positive constant, and Newton's Second Law gives

$$m \frac{dv}{dt} = mg - cv$$

- (a) Solve this as a linear equation to show that

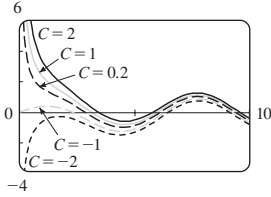
$$v = \frac{mg}{c} (1 - e^{-ct/m})$$

- (b) What is the limiting velocity?
- (c) Find the distance the object has fallen after  $t$  seconds.
36. If we ignore air resistance, we can conclude that heavier objects fall no faster than lighter objects. But if we take air resistance into account, our conclusion changes. Use the expression for the velocity of a falling object in Exercise 35(a) to find  $dv/dm$  and show that heavier objects *do* fall faster than lighter ones.

**ANSWERS**

[S Click here for solutions.](#)

- 1.** No    **3.** Yes    **5.**  $y = \frac{2}{3}e^x + Ce^{-2x}$   
**7.**  $y = x^2 \ln|x| + Cx^2$     **9.**  $y = \frac{2}{3}\sqrt{x} + C/x$   
**11.**  $y = \frac{1}{2}x + Ce^{-x^2} - \frac{1}{2}e^{-x^2} \int e^{x^2} dx$   
**13.**  $u = (t^2 + 2t + 2C)/[2(t + 1)]$     **15.**  $y = -x - 1 + 3e^x$   
**17.**  $v = t^3e^{t^2} + 5e^{t^2}$     **19.**  $y = -x \cos x - x$   
**21.**  $y = \sin x + (\cos x)/x + C/x$



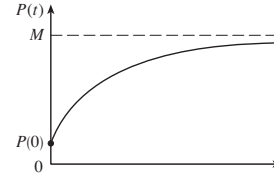
**25.**  $y = \pm[Cx^4 + 2/(5x)]^{-1/2}$

**27.** (a)  $I(t) = 4 - 4e^{-5t}$     (b)  $4 - 4e^{-1/2} \approx 1.57$  A

**29.**  $Q(t) = 3(1 - e^{-4t})$ ,  $I(t) = 12e^{-4t}$

**31.** (a) At the beginning; stays positive, but decreases

(c)  $P(t) = M + Ce^{-kt}$



**33.**  $y = \frac{2}{5}(100 + 2t) - 40,000(100 + 2t)^{-3/2}$ ; 0.2275 kg/L

**35.** (b)  $mg/c$     (c)  $(mg/c)[t + (m/c)e^{-ct/m}] - m^2g/c^2$

## SOLUTIONS

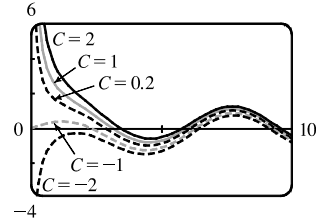
1.  $y' + e^x y = x^2 y^2$  is not linear since it cannot be put into the standard linear form (1),  $y' + P(x)y = Q(x)$ .
3.  $xy' + \ln x - x^2 y = 0 \Rightarrow xy' - x^2 y = -\ln x \Rightarrow y' + (-x)y = -\frac{\ln x}{x}$ , which is in the standard linear form (1), so this equation is linear.
5. Comparing the given equation,  $y' + 2y = 2e^x$ , with the general form,  $y' + P(x)y = Q(x)$ , we see that  $P(x) = 2$  and the integrating factor is  $I(x) = e^{\int P(x) dx} = e^{\int 2 dx} = e^{2x}$ . Multiplying the differential equation by  $I(x)$  gives  $e^{2x} y' + 2e^{2x} y = 2e^{3x} \Rightarrow (e^{2x} y)' = 2e^{3x} \Rightarrow e^{2x} y = \int 2e^{3x} dx \Rightarrow e^{2x} y = \frac{2}{3} e^{3x} + C \Rightarrow y = \frac{2}{3} e^x + C e^{-2x}$ .
7.  $xy' - 2y = x^2$  [divide by  $x$ ]  $\Rightarrow y' + \left(-\frac{2}{x}\right)y = x$  (\*).  
 $I(x) = e^{\int P(x) dx} = e^{\int (-2/x) dx} = e^{-2 \ln|x|} = e^{\ln|x|^{-2}} = e^{\ln(1/x^2)} = 1/x^2$ . Multiplying the differential equation (\*) by  $I(x)$  gives  $\frac{1}{x^2} y' - \frac{2}{x^3} y = \frac{1}{x} \Rightarrow \left(\frac{1}{x^2} y\right)' = \frac{1}{x} \Rightarrow \frac{1}{x^2} y = \ln|x| + C \Rightarrow y = x^2(\ln|x| + C) = x^2 \ln|x| + Cx^2$ .
9. Since  $P(x)$  is the derivative of the coefficient of  $y'$  [ $P(x) = 1$  and the coefficient is  $x$ ], we can write the differential equation  $xy' + y = \sqrt{x}$  in the easily integrable form  $(xy)' = \sqrt{x} \Rightarrow xy = \frac{2}{3} x^{3/2} + C \Rightarrow y = \frac{2}{3} \sqrt{x} + C/x$ .
11.  $I(x) = e^{\int 2x dx} = e^{x^2}$ . Multiplying the differential equation  $y' + 2xy = x^2$  by  $I(x)$  gives  $e^{x^2} y' + 2xe^{x^2} y = x^2 e^{x^2} \Rightarrow (e^{x^2} y)' = x^2 e^{x^2}$ . Thus  $y = e^{-x^2} \left[ \int x^2 e^{x^2} dx + C \right] = e^{-x^2} \left[ \frac{1}{2} x e^{x^2} - \int \frac{1}{2} e^{x^2} dx + C \right] = \frac{1}{2} x + C e^{-x^2} - e^{-x^2} \int \frac{1}{2} e^{x^2} dx$ .
13.  $(1+t) \frac{du}{dt} + u = 1+t, t > 0$  [divide by  $1+t$ ]  $\Rightarrow \frac{du}{dt} + \frac{1}{1+t} u = 1$  (\*), which has the form  $u' + P(t)u = Q(t)$ . The integrating factor is  $I(t) = e^{\int P(t) dt} = e^{\int [1/(1+t)] dt} = e^{\ln(1+t)} = 1+t$ . Multiplying (\*) by  $I(t)$  gives us our original equation back. We rewrite it as  $[(1+t)u]' = 1+t$ . Thus,  $(1+t)u = \int (1+t) dt = t + \frac{1}{2} t^2 + C \Rightarrow u = \frac{t + \frac{1}{2} t^2 + C}{1+t}$  or  $u = \frac{t^2 + 2t + 2C}{2(t+1)}$ .
15.  $y' = x + y \Rightarrow y' + (-1)y = x$ .  $I(x) = e^{\int (-1) dx} = e^{-x}$ . Multiplying by  $e^{-x}$  gives  $e^{-x} y' - e^{-x} y = x e^{-x} \Rightarrow (e^{-x} y)' = x e^{-x} \Rightarrow e^{-x} y = \int x e^{-x} dx = -x e^{-x} - e^{-x} + C$  [integration by parts with  $u = x, dv = e^{-x} dx$ ]  $\Rightarrow y = -x - 1 + C e^x$ .  $y(0) = 2 \Rightarrow -1 + C = 2 \Rightarrow C = 3$ , so  $y = -x - 1 + 3e^x$ .
17.  $\frac{dv}{dt} - 2tv = 3t^2 e^{t^2}, v(0) = 5$ .  $I(t) = e^{\int (-2t) dt} = e^{-t^2}$ . Multiply the differential equation by  $I(t)$  to get  $e^{-t^2} \frac{dv}{dt} - 2te^{-t^2} v = 3t^2 \Rightarrow (e^{-t^2} v)' = 3t^2 \Rightarrow e^{-t^2} v = \int 3t^2 dt = t^3 + C \Rightarrow v = t^3 e^{t^2} + C e^{t^2}$ .  $5 = v(0) = 0 \cdot 1 + C \cdot 1 = C$ , so  $v = t^3 e^{t^2} + 5e^{t^2}$ .
19.  $xy' = y + x^2 \sin x \Rightarrow y' - \frac{1}{x} y = x \sin x$ .  $I(x) = e^{\int (-1/x) dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$ . Multiplying by  $\frac{1}{x}$  gives  $\frac{1}{x} y' - \frac{1}{x^2} y = \sin x \Rightarrow \left(\frac{1}{x} y\right)' = \sin x \Rightarrow \frac{1}{x} y = -\cos x + C \Rightarrow y = -x \cos x + Cx$ .  $y(\pi) = 0 \Rightarrow -\pi \cdot (-1) + C\pi = 0 \Rightarrow C = -1$ , so  $y = -x \cos x - x$ .



21.  $y' + \frac{1}{x}y = \cos x$  ( $x \neq 0$ ), so  $I(x) = e^{\int(1/x)dx} = e^{\ln|x|} = x$  (for  $x > 0$ ). Multiplying the differential equation by  $I(x)$  gives

$$xy' + y = x \cos x \Rightarrow (xy)' = x \cos x. \text{ Thus,}$$

$$\begin{aligned} y &= \frac{1}{x} \left[ \int x \cos x \, dx + C \right] = \frac{1}{x} [x \sin x + \cos x + C] \\ &= \sin x + \frac{\cos x}{x} + \frac{C}{x} \end{aligned}$$



The solutions are asymptotic to the  $y$ -axis (except for  $C = -1$ ). In fact, for  $C > -1$ ,  $y \rightarrow \infty$  as  $x \rightarrow 0^+$ , whereas for  $C < -1$ ,  $y \rightarrow -\infty$  as  $x \rightarrow 0^+$ . As  $x$  gets larger, the solutions approximate  $y = \sin x$  more closely. The graphs for larger  $C$  lie above those for smaller  $C$ . The distance between the graphs lessens as  $x$  increases.

23. Setting  $u = y^{1-n}$ ,  $\frac{du}{dx} = (1-n)y^{-n} \frac{dy}{dx}$  or  $\frac{dy}{dx} = \frac{y^n}{1-n} \frac{du}{dx} = \frac{u^{n/(1-n)}}{1-n} \frac{du}{dx}$ . Then the Bernoulli differential equation becomes  $\frac{u^{n/(1-n)}}{1-n} \frac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$  or  $\frac{du}{dx} + (1-n)P(x)u = Q(x)(1-n)$ .

25.  $y' + \frac{2}{x}y = \frac{y^3}{x^2}$ . Here  $n = 3$ ,  $P(x) = \frac{2}{x}$ ,  $Q(x) = \frac{1}{x^2}$  and setting  $u = y^{-2}$ ,  $u$  satisfies  $u' - \frac{4u}{x} = -\frac{2}{x^2}$ .

$$\text{Then } I(x) = e^{\int(-4/x)dx} = x^{-4} \text{ and } u = x^4 \left( \int -\frac{2}{x^6} dx + C \right) = x^4 \left( \frac{2}{5x^5} + C \right) = Cx^4 + \frac{2}{5x}.$$

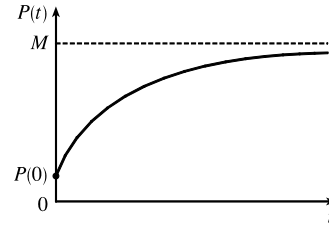
$$\text{Thus, } y = \pm \left( Cx^4 + \frac{2}{5x} \right)^{-1/2}.$$

27. (a)  $2 \frac{dI}{dt} + 10I = 40$  or  $\frac{dI}{dt} + 5I = 20$ . Then the integrating factor is  $e^{\int 5 dt} = e^{5t}$ . Multiplying the differential equation by the integrating factor gives  $e^{5t} \frac{dI}{dt} + 5Ie^{5t} = 20e^{5t} \Rightarrow (e^{5t}I)' = 20e^{5t} \Rightarrow I(t) = e^{-5t} [\int 20e^{5t} dt + C] = 4 + Ce^{-5t}$ . But  $0 = I(0) = 4 + C$ , so  $I(t) = 4 - 4e^{-5t}$ .
- (b)  $I(0.1) = 4 - 4e^{-0.5} \approx 1.57$  A

29.  $5 \frac{dQ}{dt} + 20Q = 60$  with  $Q(0) = 0$ . Then the integrating factor is  $e^{\int 4 dt} = e^{4t}$ , and multiplying the differential equation by the integrating factor gives  $e^{4t} \frac{dQ}{dt} + 4e^{4t}Q = 12e^{4t} \Rightarrow (e^{4t}Q)' = 12e^{4t} \Rightarrow Q(t) = e^{-4t} [\int 12e^{4t} dt + C] = 3 + Ce^{-4t}$ . But  $0 = Q(0) = 3 + C$  so  $Q(t) = 3(1 - e^{-4t})$  is the charge at time  $t$  and  $I = dQ/dt = 12e^{-4t}$  is the current at time  $t$ .

31. (a)  $P$  increases most rapidly at the beginning, since there are usually many simple, easily-learned sub-skills associated with learning a skill. As  $t$  increases, we would expect  $dP/dt$  to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.
- (b)  $\frac{dP}{dt} = k(M - P)$  is always positive, so the level of performance  $P$  is increasing. As  $P$  gets close to  $M$ ,  $dP/dt$  gets close to 0; that is, the performance levels off, as explained in part (a).

- (c)  $\frac{dP}{dt} + kP = kM$ , so  $I(t) = e^{\int k dt} = e^{kt}$ . Multiplying the differential equation by  $I(t)$  gives  $e^{kt} \frac{dP}{dt} + kPe^{kt} = kMe^{kt} \Rightarrow (e^{kt}P)' = kMe^{kt} \Rightarrow P(t) = e^{-kt} \left( \int kMe^{kt} dt + C \right) = M + Ce^{-kt}$ ,  $k > 0$ . Furthermore, it is reasonable to assume that  $0 \leq P(t) \leq M$ , so  $-M \leq C \leq 0$ .



33.  $y(0) = 0$  kg. Salt is added at a rate of  $\left(0.4 \frac{\text{kg}}{\text{L}}\right) \left(5 \frac{\text{L}}{\text{min}}\right) = 2 \frac{\text{kg}}{\text{min}}$ . Since solution is drained from the tank at a rate of  $3 \text{ L/min}$ , but salt solution is added at a rate of  $5 \text{ L/min}$ , the tank, which starts out with  $100 \text{ L}$  of water, contains  $(100 + 2t) \text{ L}$  of liquid after  $t$  min. Thus, the salt concentration at time  $t$  is  $\frac{y(t)}{100 + 2t} \frac{\text{kg}}{\text{L}}$ . Salt therefore leaves the tank at a rate of  $\left(\frac{y(t)}{100 + 2t} \frac{\text{kg}}{\text{L}}\right) \left(3 \frac{\text{L}}{\text{min}}\right) = \frac{3y}{100 + 2t} \frac{\text{kg}}{\text{min}}$ . Combining the rates at which salt enters and leaves the tank, we get  $\frac{dy}{dt} = 2 - \frac{3y}{100 + 2t}$ . Rewriting this equation as  $\frac{dy}{dt} + \left(\frac{3}{100 + 2t}\right)y = 2$ , we see that it is linear.  $I(t) = \exp\left(\int \frac{3 dt}{100 + 2t}\right) = \exp\left(\frac{3}{2} \ln(100 + 2t)\right) = (100 + 2t)^{3/2}$ . Multiplying the differential equation by  $I(t)$  gives  $(100 + 2t)^{3/2} \frac{dy}{dt} + 3(100 + 2t)^{1/2}y = 2(100 + 2t)^{3/2} \Rightarrow \left[(100 + 2t)^{3/2}y\right]' = 2(100 + 2t)^{3/2} \Rightarrow (100 + 2t)^{3/2}y = \frac{2}{5}(100 + 2t)^{5/2} + C \Rightarrow y = \frac{2}{5}(100 + 2t) + C(100 + 2t)^{-3/2}$ . Now  $0 = y(0) = \frac{2}{5}(100) + C \cdot 100^{-3/2} = 40 + \frac{1}{1000}C \Rightarrow C = -40,000$ , so  $y = \left[\frac{2}{5}(100 + 2t) - 40,000(100 + 2t)^{-3/2}\right] \text{ kg}$ . From this solution (no pun intended), we calculate the salt concentration at time  $t$  to be  $C(t) = \frac{y(t)}{100 + 2t} = \left[\frac{-40,000}{(100 + 2t)^{5/2}} + \frac{2}{5}\right] \frac{\text{kg}}{\text{L}}$ . In particular,  $C(20) = \frac{-40,000}{140^{5/2}} + \frac{2}{5} \approx 0.2275 \frac{\text{kg}}{\text{L}}$  and  $y(20) = \frac{2}{5}(140) - 40,000(140)^{-3/2} \approx 31.85 \text{ kg}$ .

35. (a)  $\frac{dv}{dt} + \frac{c}{m}v = g$  and  $I(t) = e^{\int (c/m) dt} = e^{(c/m)t}$ , and multiplying the differential equation by  $I(t)$  gives  $e^{(c/m)t} \frac{dv}{dt} + \frac{vce^{(c/m)t}}{m} = ge^{(c/m)t} \Rightarrow \left[e^{(c/m)t}v\right]' = ge^{(c/m)t}$ . Hence,  $v(t) = e^{-(c/m)t} \left[\int ge^{(c/m)t} dt + K\right] = mg/c + Ke^{-(c/m)t}$ . But the object is dropped from rest, so  $v(0) = 0$  and  $K = -mg/c$ . Thus, the velocity at time  $t$  is  $v(t) = (mg/c) \left[1 - e^{-(c/m)t}\right]$ .
- (b)  $\lim_{t \rightarrow \infty} v(t) = mg/c$
- (c)  $s(t) = \int v(t) dt = (mg/c) \left[t + (m/c)e^{-(c/m)t}\right] + c_1$  where  $c_1 = s(0) - m^2g/c^2$ .  $s(0)$  is the initial position, so  $s(0) = 0$  and  $s(t) = (mg/c) \left[t + (m/c)e^{-(c/m)t}\right] - m^2g/c^2$ .