Double Integrals

The definite integral of a continuous function f of one variable on an interval $[a, b]$ is defined as

$$\boxed{1} \qquad \int_a^b f(x)\,dx = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

where $\Delta x = (b - a)/n$ and $x_1, x_2, \ldots, x_n$ are the endpoints of the subintervals of $[a, b]$ with width Δx. We saw that if $f(x)$ is a positive function, then $\int_a^b f(x)\,dx$ can be interpreted as an area and we used $\int_a^b f(x)\,dx$ to compute average values.

In a similar way we will show here how to define the double integral of a function of two variables $f(x, y)$ on a rectangle. We will see how to interpret it as a volume if $f(x, y)$ is a positive function and how to use it to calculate average values.

Double Integrals over Rectangles

We start with a function $f(x, y)$ whose domain is a rectangle

$$R = \{(x, y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

If we divide the interval $[a, b]$ into m subintervals of equal width $\Delta x = (b - a)/m$ and we divide the interval $[c, d]$ into n subintervals of equal width $\Delta y = (d - c)/n$, then, as shown in Figure 1, R is divided into mn subrectangles each with area $\Delta A = \Delta x\,\Delta y$. The upper right corner of a typical subrectangle has coordinates (x_i, y_j).

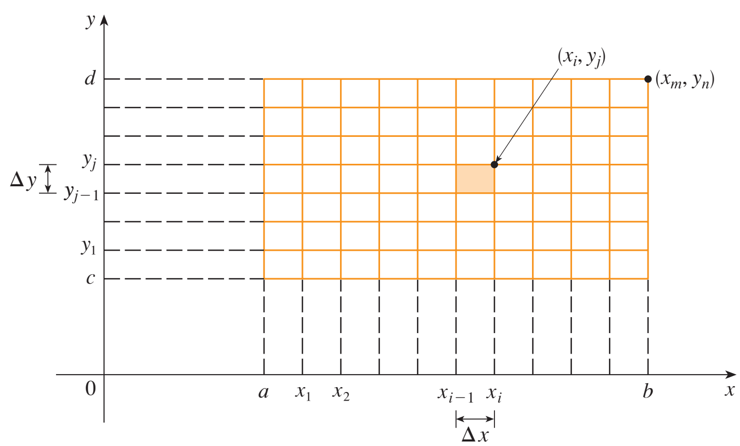


FIGURE 1
Dividing R into subrectangles

By analogy with Equation 1 we define the **double integral** of f over the rectangle R as a limit of double Riemann sums:

$$\boxed{2} \qquad \iint\limits_R f(x, y)\,dA = \lim_{m,n\to\infty} [f(x_1, y_1)\,\Delta A + f(x_1, y_2)\,\Delta A + \cdots + f(x_m, y_n)\,\Delta A]$$

There are a total of mn terms in the Riemann sum in Definition 2, one for each of the mn subrectangles in Figure 1. If the limit exists, f is called *integrable*.

For a positive function we can interpret $f(x_i, y_j)$ as the height of a thin rectangular column with base area ΔA and volume $f(x_i, y_j)\,\Delta A$. (See Figure 2.) So the Riemann sum in Definition 2 can be interpreted as the sum of volumes of columns (see Figure 3) and this sum is an approximation to the volume of the solid that lies under the graph of the surface $z = f(x, y)$ and above the rectangle R.

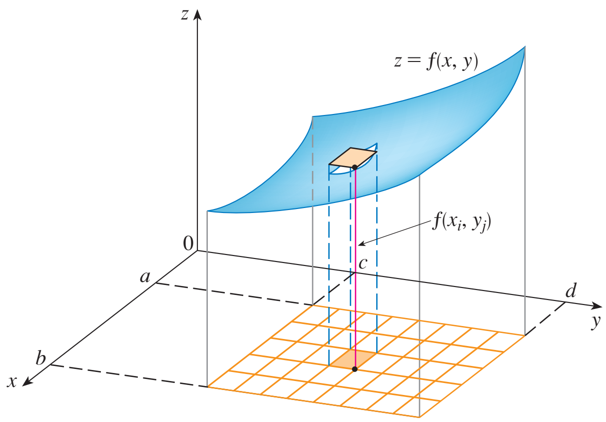


FIGURE 2

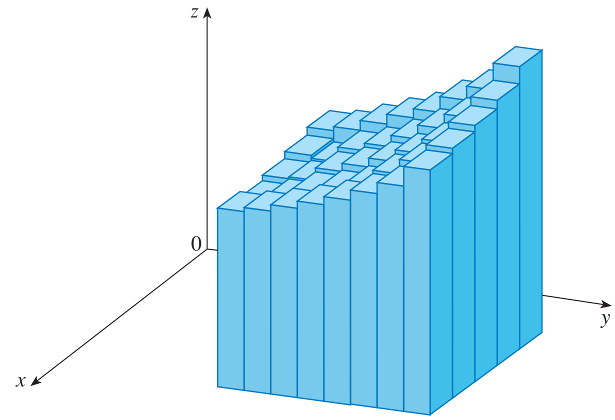


FIGURE 3

As m and n become large in Definition 2 and Figures 2 and 3, the approximation becomes closer and closer to the actual volume, so we define the *volume* of the solid to be the value of the double integral.

If $f(x, y) \geqslant 0$, the **volume** of the solid that lies under the surface $z = f(x, y)$ and above the rectangle R is

$$V = \iint_R f(x, y)\, dA$$

Iterated Integrals

It's very difficult to evaluate a double integral using Definition 2 directly, so now we show how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed (and treated as a constant) and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y)\, dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\, dy$$

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

3 $$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

The integral on the right side of Equation 7 is called an **iterated integral**. Usually the brackets are omitted. Thus

4 $$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

5 $$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$. Notice that in both Equations 8 and 9 we work *from the inside out.*

EXAMPLE 1 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

SOLUTION

(a) Working from the inside out, we first evaluate $\int_1^2 x^2 y \, dy$. Regarding x as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \tfrac{3}{2} x^2$$

Thus the function A in the preceding discussion is given by $A(x) = \frac{3}{2} x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2 \, dx = \frac{x^3}{2} \Big]_0^3 = \frac{27}{2}$$

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy = 9 \frac{y^2}{2} \Big]_1^2 = \frac{27}{2}$$ ■

Notice that in Example 1 we obtained the same answer whether we integrated with respect to y or x first. In general, it turns out (see Theorem 6) that the two iterated integrals in Equations 4 and 5 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

The practical method for evaluating a *double* integral is to express it as an *iterated* integral (in either order). The following theorem is true for most functions that one meets in practice. It is proved in courses on advanced calculus.

6 If $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

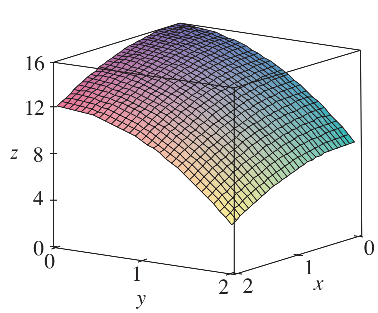


FIGURE 4

EXAMPLE 2 Find the volume of the solid S that is enclosed by a paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

SOLUTION

We first observe that S is the solid that lies under the surface $z = 16 - x^2 - 2y^2$ and above the square $R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 2\}$. (See Figure 4.) Using Theorem 6

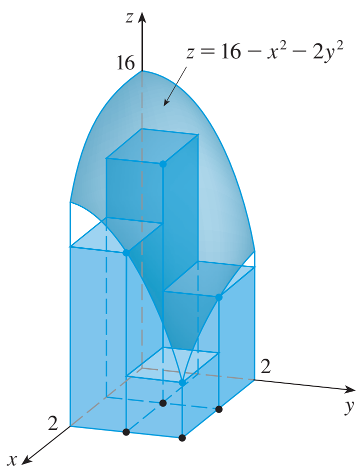


FIGURE 5

we express the double integral for the volume as an iterated integral:

$$\begin{aligned} V &= \iint_R (16 - x^2 - 2y^2)\, dA \\ &= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy \\ &= \int_0^2 \left[16x - \tfrac{1}{3}x^3 - 2y^2x\right]_{x=0}^{x=2} dy \\ &= \int_0^2 \left(\tfrac{88}{3} - 4y^2\right) dy = \left[\tfrac{88}{3}y - \tfrac{4}{3}y^3\right]_0^2 = 48 \end{aligned}$$ ■

NOTE Figure 5 illustrates the definitions of volume and the double integral by showing how the solid in Example 2 is approximated by the four columns in the Riemann sum with $m = n = 2$. Figure 6 shows how the columns become better approximations to the volume as m and n increase.

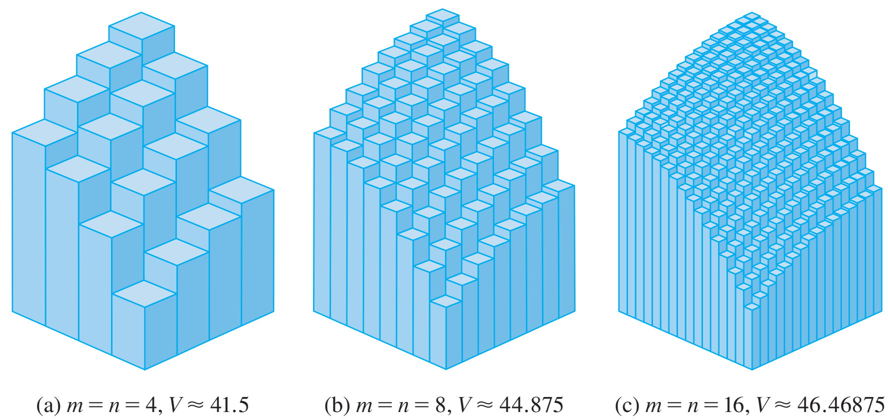


FIGURE 6 The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increases.

Double Integrals over More General Regions

What happens if we need to integrate a function $f(x, y)$ over a region D that is not a rectangle? Suppose, for instance, that the domain D of f lies between the graphs of two continuous functions of x:

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

Figure 7 shows three examples of such regions.

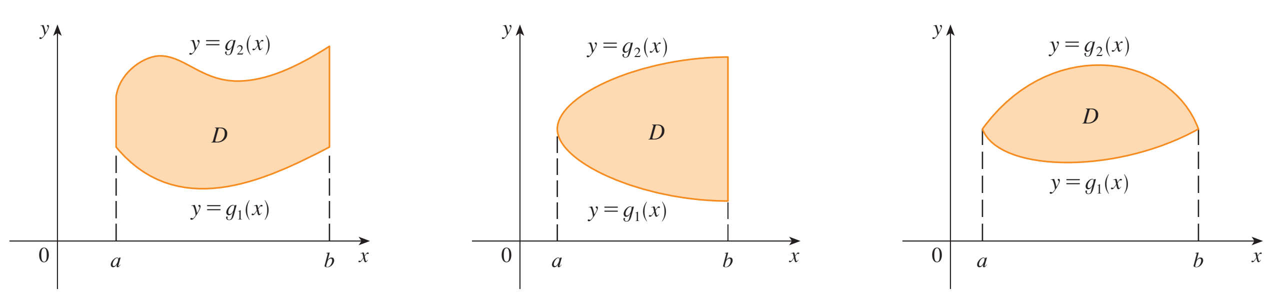


FIGURE 7

The double integral of f over D, $\iint_D f(x, y)\, dA$, can be defined by a limit similar to the one in Definition 2 and it can be evaluated as an iterated integral similar to the one in Theorem 6:

7
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Notice that the lower and upper limits of integration in the inner integral in Equation 7 are functions of x: $y = g_1(x)$ and $y = g_2(x)$. This makes sense because for a fixed value of x between a and b, y goes from the lower boundary curve $y = g_1(x)$ to the upper boundary curve $y = g_2(x)$. But in evaluating the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$. Notice also that in the special case where $g_1(x) = c$ and $g_2(x) = d$, D is a rectangle and Equation 7 is the same as the first part of Theorem 6.

EXAMPLE 3 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

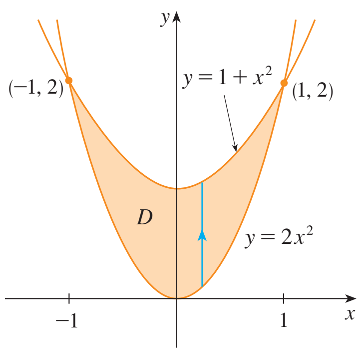


FIGURE 8

SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. The region D is sketched in Figure 8 and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 7 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}$$ ■

NOTE When we set up a double integral as in Example 3, it is essential to draw a diagram. Often it is helpful to draw a vertical arrow as in Figure 8. Then the limits of integration for the *inner* integral can be read from the diagram as follows: The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral, and the arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration.

Average Value

Recall from Section 6.2 that the average value of a function f of one variable defined on an interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

In a similar fashion we define the **average value** of a function f of two variables defined on a rectangle R to be

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where $A(R)$ is the area of R.

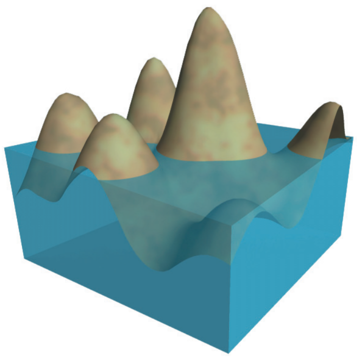


FIGURE 9

If $f(x, y) \geqslant 0$, the equation

$$A(R) \times f_{\text{ave}} = \iint_R f(x, y)\, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f. [If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{ave}, then you can use them to fill in the valleys so that the region becomes completely flat. See Figure 9.]

EXAMPLE 4 A manufacturer has modeled its output by a Cobb-Douglas production function

$$P(L, K) = 70L^{0.6}K^{0.4}$$

where L is the number of monthly labor hours and K is the monthly capital investment (in units of \$1000). If L varies roughly evenly from 5000 to 6000 and monthly capital investment varies evenly between \$20,000 and \$30,000, find the average monthly output.

SOLUTION We compute the average value of the function $P(L, K)$ over the rectangular region R defined by

$$5000 \leqslant L \leqslant 6000 \qquad 20 \leqslant K \leqslant 30$$

The area of R is

$$A(R) = (6000 - 5000)(30 - 20) = 10{,}000$$

So the average value is

$$\begin{aligned}
P_{\text{ave}} &= \frac{1}{A(R)} \iint_R P(L, K)\, dA \\
&= \frac{1}{10{,}000} \int_{5000}^{6000} \int_{20}^{30} P(L, K)\, dK\, dL \\
&= \frac{1}{10{,}000} \int_{5000}^{6000} \int_{20}^{30} 70L^{0.6}K^{0.4}\, dK\, dL \\
&= \frac{1}{10{,}000} \int_{5000}^{6000} \left[70L^{0.6} \frac{K^{1.4}}{1.4} \right]_{K=20}^{K=30} dL \\
&= \frac{1}{10{,}000} \int_{5000}^{6000} 50L^{0.6}(30^{1.4} - 20^{1.4})\, dL \\
&= \frac{(30^{1.4} - 20^{1.4})}{200} \left[\frac{L^{1.6}}{1.6} \right]_{5000}^{6000} \\
&= \frac{(30^{1.4} - 20^{1.4})(6000^{1.6} - 5000^{1.6})}{320} \approx 44{,}427.0
\end{aligned}$$

The average monthly output is about 44,427 units. ■

Exercises

1–2 ■ Find $\int_0^5 f(x, y)\,dx$ and $\int_0^1 f(x, y)\,dy$.

1. $f(x, y) = 12x^2y^3$

2. $f(x, y) = y + xe^y$

3–12 ■ Calculate the iterated integral.

3. $\int_1^3 \int_0^1 (1 + 4xy)\,dx\,dy$

4. $\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\,dy\,dx$

5. $\int_0^2 \int_0^1 (2x + y)^8\,dx\,dy$

6. $\int_0^1 \int_1^2 \frac{xe^x}{y}\,dy\,dx$

7. $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right)dy\,dx$

8. $\int_0^1 \int_0^1 \sqrt{s + t}\,ds\,dt$

9. $\int_0^4 \int_0^{\sqrt{y}} xy^2\,dx\,dy$

10. $\int_0^1 \int_{2x}^2 (x - y)\,dy\,dx$

11. $\int_0^1 \int_{x^2}^x (1 + 2y)\,dy\,dx$

12. $\int_0^2 \int_y^{2y} xy\,dx\,dy$

13–20 ■ Calculate the double integral.

13. $\iint_R (6x^2y^3 - 5y^4)\,dA, \quad R = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 1\}$

14. $\iint_R (y + xy^{-2})\,dA, \quad R = \{(x, y) \mid 0 \le x \le 2, 1 \le y \le 2\}$

15. $\iint_R xye^{x^2y}\,dA, \quad R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2\}$

16. $\iint_R \frac{x}{1 + xy}\,dA, \quad R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$

17. $\iint_R xy^2\,dA$, D is the triangular region with vertices (0, 0), (1, 0), and (1, 1)

18. $\iint_D \frac{y}{x^5 + 1}\,dA, \quad D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le x^2\}$

19. $\iint_D x^3\,dA, \quad D = \{(x, y) \mid 1 \le x \le e, 0 \le y \le \ln x\}$

20. $\iint_D (x + y)\,dA$, D is bounded by $y = \sqrt{x}$ and $y = x^2$

21. Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R = \{(x, y) \mid 0 \le x \le 1, -2 \le y \le 3\}$.

22. Find the volume of the solid that lies under the surface $z = 4 + x^2 - y^2$ and above the square $R = \{(x, y) \mid -1 \le x \le 1, 0 \le y \le 2\}$.

23. Find the volume of the solid that lies under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1), and (1, 2).

24. Find the volume of the solid that is enclosed by the coordinate planes and the plane $3x + 2y + z = 6$.

25–27 ■ Find the average value of f over the given region.

25. $f(x, y) = x^2y$, R is the rectangle with vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, $(1, 0)$

26. $f(x, y) = e^y\sqrt{x + e^y}, \quad R = \{(x, y) \mid 0 \le x \le 4, 0 \le y \le 1\}$

27. $f(x, y) = xy$, D is the triangle with vertices (0, 0), (1, 0), and (1, 3)

28. A company models its monthly production by the function

$$P(x, y) = 200x^{3/4}y^{1/4}$$

where x is the number of workers and y is the monthly operating budget in thousand of dollars. The company uses between 50 and 60 workers and its operating budget varies from \$40,000 to \$50,000 per month. Estimate the average monthly output.

29. The state of Colorado is in the shape of a rectangle that measures 388 miles west to east and 276 miles south to north. Suppose the function

$$f(x, y) = 4.6 + 0.02x - 0.01y + 0.0001xy$$

approximates the snowfall, in inches, left during a storm at a location x miles east and y miles north of the southwest corner of the state. According to the model, what was the average snowfall for the entire state during the storm?

30. Researchers assessed the level of airborne pollution, in parts per million (ppm), created by a manufacturing facility throughout a nearby rectangular plot of farmland. The boundaries of the land run 4 miles from east to west and 2 miles from south to north. The level of pollution, measured in parts per million (ppm), at a location x miles west and y miles north of the southeast corner of the plot was modeled by $f(x, y) = 2.7e^{-0.1x-0.4y}$. Use the model to find the average pollution level over the entire plot of land.

Answers

1. $500y^3, 3x^2$ **3.** 10 **5.** 261,632/45 **7.** $\frac{21}{2} \ln 2$

9. 32 **11.** $\frac{3}{10}$ **13.** $\frac{21}{2}$ **15.** $\frac{1}{2}(e^2 - 3)$ **17.** $\frac{1}{15}$

19. $\frac{3}{16}e^4 + \frac{1}{16}$ **21.** 47.5 **23.** $\frac{31}{8}$ **25.** $\frac{5}{6}$ **27.** $\frac{3}{4}$

29. About 9.8 inches

Solutions

1. $\int_0^5 12x^2y^3\,dx = \left[12\,\dfrac{x^3}{3}\,y^3\right]_{x=0}^{x=5} = 4x^3y^3\big]_{x=0}^{x=5} = 4(5)^3\,y^3 - 4(0)^3\,y^3 = 500y^3,$

$\int_0^1 12x^2y^3\,dy = \left[12x^2\,\dfrac{y^4}{4}\right]_{y=0}^{y=1} = 3x^2y^4\big]_{y=0}^{y=1} = 3x^2(1)^4 - 3x^2(0)^4 = 3x^2$

3. $\int_1^3\int_0^1(1+4xy)\,dx\,dy = \int_1^3\left[x+2x^2y\right]_{x=0}^{x=1}dy = \int_1^3[(1+2y)-(0+0)]\,dy = \int_1^3(1+2y)\,dy = \left[y+y^2\right]_1^3$

$= (3+9)-(1+1) = 10$

5. $\int_0^2\int_0^1(2x+y)^8dx\,dy = \displaystyle\int_0^2\left[\frac{1}{2}\,\frac{(2x+y)^9}{9}\right]_{x=0}^{x=1}dy$ [substitute $u = 2x+y \;\Rightarrow\; dx = \frac{1}{2}\,du$]

$= \dfrac{1}{18}\displaystyle\int_0^2[(2+y)^9-(0+y)^9]\,dy = \frac{1}{18}\left[\frac{(2+y)^{10}}{10}-\frac{y^{10}}{10}\right]_0^2$

$= \frac{1}{180}[(4^{10}-2^{10})-(2^{10}-0^{10})] = \frac{1{,}046{,}528}{180} = \frac{261{,}632}{45}$

7. $\displaystyle\int_1^4\int_1^2\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx = \int_1^4\left[x\ln|y|+\frac{1}{x}\cdot\frac{1}{2}\,y^2\right]_{y=1}^{y=2}dx = \int_1^4\left(x\ln 2+\frac{2}{x}-0-\frac{1}{2x}\right)dx = \int_1^4\left(x\ln 2+\frac{3}{2x}\right)dx$

$= \left[\frac{1}{2}x^2\ln 2+\frac{3}{2}\ln|x|\right]_1^4 = 8\ln 2+\frac{3}{2}\ln 4-\frac{1}{2}\ln 2 = \frac{15}{2}\ln 2+3\ln 4^{1/2} = \frac{21}{2}\ln 2$

9. $\int_0^4\int_0^{\sqrt{y}}xy^2\,dx\,dy = \int_0^4\left[\frac{1}{2}x^2y^2\right]_{x=0}^{x=\sqrt{y}}dy = \int_0^4\frac{1}{2}y^2[(\sqrt{y}\,)^2-0^2]dy = \frac{1}{2}\int_0^4 y^3\,dy = \frac{1}{2}\left[\frac{1}{4}y^4\right]_0^4 = \frac{1}{2}(64-0) = 32$

11. $\int_0^1\int_{x^2}^x(1+2y)\,dy\,dx = \int_0^1\left[y+y^2\right]_{y=x^2}^{y=x}dx = \int_0^1\left[x+x^2-x^2-(x^2)^2\right]dx = \int_0^1(x-x^4)dx$

$= \left[\frac{1}{2}x^2-\frac{1}{5}x^5\right]_0^1 = \frac{1}{2}-\frac{1}{5}-0+0 = \frac{3}{10}$

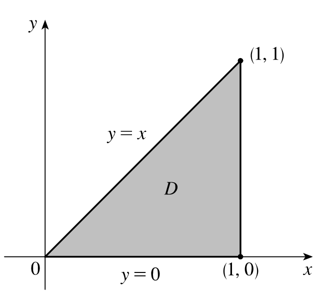
13. $\iint_R(6x^2y^3-5y^4)\,dA = \int_0^3\int_0^1(6x^2y^3-5y^4)\,dy\,dx = \int_0^3\left[\frac{3}{2}x^2y^4-y^5\right]_{y=0}^{y=1}dx = \int_0^3\left(\frac{3}{2}x^2-1\right)dx$

$= \left[\frac{1}{2}x^3-x\right]_0^3 = \frac{27}{2}-3 = \frac{21}{2}$

15. $\iint_R xye^{x^2y}\,dA = \int_0^2\int_0^1 xye^{x^2y}\,dx\,dy = \int_0^2\left[\frac{1}{2}e^{x^2y}\right]_{x=0}^{x=1}dy$ [let $u = x^2y \;\Rightarrow\; du = 2xy\,dx$]

$= \frac{1}{2}\int_0^2(e^y-1)\,dy = \frac{1}{2}\left[e^y-y\right]_0^2 = \frac{1}{2}[(e^2-2)-(1-0)] = \frac{1}{2}(e^2-3)$

17.



D is the region below the line $y = x$ and above the line $y = 0$ for $0 \le x \le 1$, so $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\}$ and

$\displaystyle\iint_D xy^2\,dA = \int_0^1\int_0^x xy^2\,dy\,dx = \int_0^1 x\left[\frac{1}{3}y^3\right]_{y=0}^{y=x}dx$

$\displaystyle= \int_0^1\frac{1}{3}x^4\,dx = \left[\frac{1}{3}\cdot\frac{1}{5}x^5\right]_0^1 = \frac{1}{15}-0 = \frac{1}{15}$

19. $\iint_D x^3\,dA = \int_1^e\int_0^{\ln x}x^3\,dy\,dx = \int_1^e\left[x^3y\right]_{y=0}^{y=\ln x}dx = \int_1^e x^3\ln x\,dx$ [integrate by parts with $u = \ln x$, $dv = x^3\,dx$]

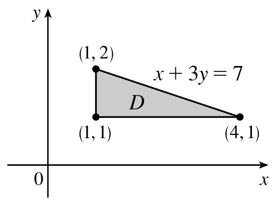
$= \left[\frac{1}{4}x^4\ln x-\frac{1}{16}x^4\right]_1^e = \frac{1}{4}e^4-\frac{1}{16}e^4-0+\frac{1}{16} = \frac{3}{16}e^4+\frac{1}{16}$

21. The plane $3x + 2y + z = 12$ is the function $z = 12 - 3x - 2y$, so the volume of the solid is

$$V = \iint_R (12 - 3x - 2y)\,dA = \int_{-2}^{3}\int_0^1 (12 - 3x - 2y)\,dx\,dy = \int_{-2}^{3}\left[12x - \tfrac{3}{2}x^2 - 2xy\right]_{x=0}^{x=1} dy$$

$$= \int_{-2}^{3}\left(\tfrac{21}{2} - 2y\right) dy = \left[\tfrac{21}{2}y - y^2\right]_{-2}^{3} = \tfrac{95}{2}$$

23. The region of integration D is the region below the line $x + 3y = 7 \quad \Leftrightarrow \quad y = (7 - x)/3$ and above the line $y = 1$ for $1 \le x \le 4$. Thus



$$V = \int_1^4 \int_1^{(7-x)/3} xy\,dy\,dx = \int_1^4 \left[\tfrac{1}{2}xy^2\right]_{y=1}^{y=(7-x)/3} dx$$

$$= \int_1^4 \tfrac{1}{2}x\left[\tfrac{1}{9}(7-x)^2 - 1\right] dx = \tfrac{1}{2}\int_1^4 \tfrac{1}{9}x\left[(7-x)^2 - 9\right] dx$$

$$= \tfrac{1}{18}\int_1^4 (x^3 - 14x^2 + 40x)\,dx = \tfrac{1}{18}\left[\tfrac{1}{4}x^4 - \tfrac{14}{3}x^3 + 20x^2\right]_1^4$$

$$= \tfrac{1}{18}\left(\tfrac{256}{3} - \tfrac{187}{12}\right) = \tfrac{31}{8}$$

25. R is the rectangle $[-1, 1] \times [0, 5]$. Thus, $A(R) = 2 \cdot 5 = 10$ and

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \tfrac{1}{10}\int_0^5\int_{-1}^1 x^2y\,dx\,dy = \tfrac{1}{10}\int_0^5\left[\tfrac{1}{3}x^3y\right]_{x=-1}^{x=1} dy = \tfrac{1}{10}\int_0^5 \tfrac{2}{3}y\,dy = \tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5 = \tfrac{5}{6}.$$

27. The average value of a function f of two variables defined on a rectangle R was defined as $f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)dA$, where $A(R)$ is the area of the region of integration R. Extending this definition to general regions D, we have

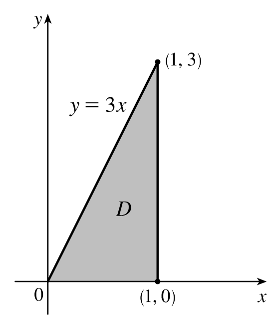
$f_{\text{ave}} = \frac{1}{A(D)}\iint_D f(x,y)dA$.

Here $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3x\}$, so $A(D) = \frac{1}{2}(1)(3) = \frac{3}{2}$

and

$$f_{\text{ave}} = \frac{1}{A(D)}\iint_D f(x,y)dA = \frac{1}{3/2}\int_0^1\int_0^{3x} xy\,dy\,dx$$

$$= \tfrac{2}{3}\int_0^1\left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=3x} dx = \tfrac{1}{3}\int_0^1 9x^3\,dx = \tfrac{3}{4}x^4\Big]_0^1 = \tfrac{3}{4}$$



29. If we place the origin at the southwest corner of the state, then the region is described by the rectangle $R = \{(x,y) \mid 0 \le x \le 388, 0 \le y \le 276\}$. The area of R is $388 \cdot 276 = 107{,}088$, and the average snowfall was

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{107{,}088}\int_0^{388}\int_0^{276}(4.6 + 0.02x - 0.01y + 0.0001xy)\,dy\,dx$$

$$= \frac{1}{107{,}088}\int_0^{388}\left[4.6y + 0.02xy - 0.005y^2 + 0.00005xy^2\right]_{y=0}^{y=276} dx$$

$$= \frac{1}{107{,}088}\int_0^{388}(1269.6 + 5.52x - 380.88 + 3.8088x)\,dx = \frac{1}{107{,}088}\int_0^{388}(888.72 + 9.3288x)\,dx$$

$$= \frac{1}{107{,}088}\left[888.72x + 4.6644x^2\right]_0^{388} = \frac{1}{107{,}088}(1{,}047{,}020.794) \approx 9.77 \text{ in}$$